show that Eq. (2-129) can be expressed as

$$R_r = \frac{\eta}{2\pi}\left[C + \log kL - \text{Ci}\, kL + \sin kL\,(\tfrac{1}{2}\text{Si}\, 2kL - \text{Si}\, kL) + \tfrac{1}{2}\cos kL\left(C + \log k\frac{L}{2} + \text{Ci}\, 2kL - 2\text{Ci}\, kL\right)\right]$$

where $C = 0.5772 \cdots$ is Euler's constant.

2-45. If the linear antenna of Fig. 2-23 is an integral number of half-wavelengths long, the current will assume the form

$$I(z) = I_m \sin k\left(z + \frac{L}{2}\right)$$

regardless of the position of the feed as long as it is not near a current null. Such an antenna is said to be of *resonant length*. Show that the radiation field of the antenna is

$$E_\theta = \frac{j\eta I_m}{2\pi r} e^{-jkr} \frac{\cos\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} \qquad n \text{ odd}$$

$$E_\theta = \frac{\eta I_m}{2\pi r} e^{-jkr} \frac{\sin\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} \qquad n \text{ even}$$

where $n = 2L/\lambda$ is an integer.

2-46. For an antenna of resonant length (Prob. 2-45), show that the radiation resistance referred to I_m is

$$R_r = \frac{\eta}{4\pi}\left[C + \log 2n\pi - \text{Ci}(2n\pi)\right]$$

where $n = 2L/\lambda$, $C = 0.5772$, and Ci is as defined in Prob. 2-44. Show that the input resistance for a loss-free antenna with feed point at $z = a\lambda$ is

$$R_i = \frac{R_r}{\sin 2\pi(a + n/4)}$$

Specialize this result to $L = \lambda/2$, $a = 0$ (the half-wave dipole) and show that $R_i = 73$ ohms.

CHAPTER 3

SOME THEOREMS AND CONCEPTS

3-1. The Source Concept. The complex field equations for linear media are

$$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H} + \mathbf{M} \qquad \nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J} \qquad (3\text{-}1)$$

where $\mathbf{J}$ and $\mathbf{M}$ are sources in the most general sense. We have purposely omitted superscripts on $\mathbf{J}$ and $\mathbf{M}$ because their interpretations vary from problem to problem. In one problem, they might represent actual sources, in which case we would call them impressed currents. In another problem, $\mathbf{J}$ might represent a conduction current that we wish to keep separate from the $\hat{y}\mathbf{E}$ term. In still another problem, $\mathbf{M}$ might represent a magnetic polarization current that we wish to keep separate from the $\hat{z}\mathbf{H}$ term, and so on. We can think of $\mathbf{J}$ and $\mathbf{M}$ as "mathematical sources," regardless of their physical interpretation.

For our first illustration, let us show how to represent "circuit sources" in terms of the "field sources" $\mathbf{J}$ and $\mathbf{M}$. The *current source* of circuit theory is defined as one whose current is independent of the load. In terms of field concepts it can be pictured as a short filament of impressed electric current in series with a perfectly conducting wire. This is shown in Fig. 3-1*a*. That it has the characteristics of the current source of circuit theory can be demonstrated as follows. We make the usual circuit assumption that the displacement current through the surrounding medium is negligible. It then follows from the conservation of charge that the current in the leads is equal to the impressed current, independent of the load. The field formula for power, Eq. (1-66), reduces to

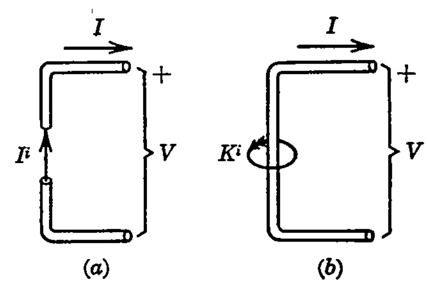

FIG. 3-1. Circuit sources in terms of impressed currents. (*a*) Current source; (*b*) voltage source.

the circuit formula for this source. We have only electric currents; hence

$$P_s = -\iiint \mathbf{E}\cdot\mathbf{J}^{i*}\,d\tau = -I^{i*}\int \mathbf{E}\cdot d\mathbf{l} = VI^*$$

The "internal impedance" of the source is infinite, since a removal of the impressed current leaves an open circuit.

The *voltage source* of circuit theory is defined as one whose voltage is independent of the load. In terms of field concepts it can be pictured as a small loop of impressed magnetic current encircling a perfectly conducting wire. This is illustrated by Fig. 3-1*b*. To show that it has the characteristics of the voltage source of circuit theory, we neglect displacement current and apply the field equation $K = -\oint \mathbf{E}\cdot d\mathbf{l}$ to a path coincident with the wire and closing across the terminals. The $\mathbf{E}$ is zero in the wire; so the line integral is merely the terminal voltage, that is, $K^i = -V$. The impressed current, and therefore the terminal voltage, is independent of load. The field formula for power, Eq. (1-66), reduces in this case to

$$P_s = -\iiint \mathbf{H}^*\cdot\mathbf{M}^i\,d\tau = -K^i\oint \mathbf{H}^*\cdot d\mathbf{l} = VI^*$$

which is the usual circuit formula. The internal impedance of the source is zero, since a removal of the impressed current leaves a short circuit.

We can use the circuit sources in field problems when the source and input region are of "circuit dimensions," that is, of dimensions small compared to a wavelength. Given a pair of terminals close together, we can apply the current source of Fig. 3-1*a*, that is, a short filament of impressed electric current. Given a conductor of small cross section, we can apply the voltage source of Fig. 3-1*b*, that is, a small loop of impressed magnetic current. As an example of the use of a circuit source, consider the linear antenna of Fig. 2-23. The geometry of the physical antenna is two sections of wire separated by a small gap at the input. To excite the antenna, we can place a current source (a short filament of electric current) across the gap, which causes a current in the antenna wire. An exact solution to the problem involves a determination of the resulting current in the wire. This is difficult to do. Instead, we approximate the current in the wire, drawing on qualitative and experimental knowledge. We then use this current, plus the current source across the gap, in the potential integral formula to give us an approximation to the field.

We shall find much use for the concept of current sheets, considered in Sec. 1-14. As an example, suppose we have a $\mathbf{J}_s$ over the cross section of a rectangular waveguide, as shown in Fig. 3-2. Furthermore, we postulate that this current should produce only the TE_{01} waveguide mode,

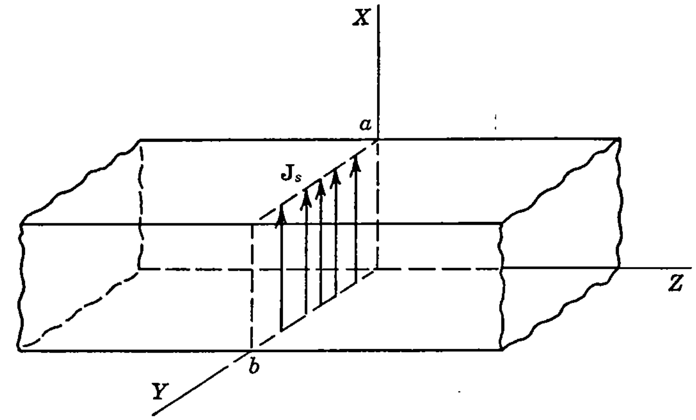

FIG. 3-2. A sheet of current in a rectangular waveguide.

which propagates outward from the current sheet. Abstracting from Table 2-4, we have the wave

$$\left.\begin{aligned} E_x^+ &= A\sin\frac{\pi y}{b}\,e^{-j\beta z} \\ H_y^+ &= \frac{A}{Z_0}\sin\frac{\pi y}{b}\,e^{-j\beta z} \\ H_z^+ &= \frac{A}{j\eta}\frac{f_c}{f}\cos\frac{\pi y}{b}\,e^{-j\beta z} \end{aligned}\right\} \quad z > 0$$

where the constant A specifies the mode amplitude. The $-z$ traveling wave is of the same form with β replaced by $-\beta$ and Z_0 by $-Z_0$. Thus,

$$\left.\begin{aligned} E_x^- &= B\sin\frac{\pi y}{b}\,e^{j\beta z} \\ H_y^- &= -\frac{B}{Z_0}\sin\frac{\pi y}{b}\,e^{j\beta z} \\ H_z^- &= \frac{B}{j\eta}\frac{f_c}{f}\cos\frac{\pi y}{b}\,e^{j\beta z} \end{aligned}\right\} \quad z < 0$$

where B is the mode amplitude of the $-z$ traveling wave. At $z = 0$, Eqs. (1-86) must be satisfied. Take the (1) side to be $z > 0$, so that $\mathbf{n} = \mathbf{u}_z$, and obtain

$$-\mathbf{u}_x[H_y^+ - H_y^-]_{z=0} = \mathbf{J}_s \qquad [E_x^+ - E_x^-]_{z=0} = 0$$

Substitution for H_y and E_x from above reduces these equations to

$$-\mathbf{u}_x\frac{A+B}{Z_0}\sin\frac{\pi y}{b} = \mathbf{J}_s \qquad A - B = 0$$

Let

$$\mathbf{J}_s = \mathbf{u}_x J_0 \sin\frac{\pi y}{b} \tag{3-2}$$

The preceding equations then have the solution $A = B = -J_0Z_0/2$. Thus, if the current of Eq. (3-2) exists over the guide cross section $z = 0$, then

$$E_x = \begin{cases} -\dfrac{J_0Z_0}{2}\sin\dfrac{\pi y}{b}\,e^{-j\beta z} & z > 0 \\ -\dfrac{J_0Z_0}{2}\sin\dfrac{\pi y}{b}\,e^{j\beta z} & z < 0 \end{cases} \tag{3-3}$$

It would admittedly be difficult to obtain the current of Eq. (3-2) in practice, but this is not of concern at present. We shall learn how to treat more practical problems later. Note that our approach in this problem was to assume the field and find the current. This we shall find to be a very powerful concept.

3-2. Duality. If the equations describing two different phenomena are of the same mathematical form, solutions to them will take the same mathematical form. The formal recognition of this is called the *concept of duality*. Two equations of the same mathematical form are called *dual equations*. Quantities occupying the same position in dual equations are called *dual quantities*. Note that the field equations, Eqs. (3-1), are duals of each other. A systematic interchange of symbols changes the first equation into the second, and vice-versa.

A duality of importance to us is that between a problem for which all sources are of the electric type and a problem for which all sources are of the magnetic type. The first two rows of Table 3-1 give the field equations in each case. The last two formulas of column (1) were derived in Sec. 2-9 for homogeneous space. The corresponding equations for the magnetic source case are evidently the last two formulas of column (2), obtained by systematically interchanging symbols. The particular interchange of symbols is summarized by Table 3-2. The reader should check for himself that a replacement of the symbols of column (1) of Table 3-2 by those of column (2) in the equations of column (1) of Table 3-1 results in the equations of column (2). The quantity **F** of these tables is called an *electric vector potential*, in analogy to **A**, a magnetic vector potential.

TABLE 3-1. DUAL EQUATIONS FOR PROBLEMS IN WHICH (1) ONLY ELECTRIC SOURCES EXIST AND (2) ONLY MAGNETIC SOURCES EXIST

(1) Electric sources	(2) Magnetic sources
$\nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J}$	$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H} + \mathbf{M}$
$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H}$	$\nabla \times \mathbf{H} = \hat{y}\mathbf{E}$
$\mathbf{H} = \nabla \times \mathbf{A}$	$\mathbf{E} = -\nabla \times \mathbf{F}$
$\mathbf{A} = \dfrac{1}{4\pi}\iiint \dfrac{\mathbf{J}e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r}-\mathbf{r}'\rvert}\,d\tau'$	$\mathbf{F} = \dfrac{1}{4\pi}\iiint \dfrac{\mathbf{M}e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r}-\mathbf{r}'\rvert}\,d\tau'$

TABLE 3-2. DUAL QUANTITIES FOR PROBLEMS IN WHICH (1) ONLY ELECTRIC SOURCES EXIST, AND (2) ONLY MAGNETIC SOURCES EXIST

(1) *Electric sources*	(2) *Magnetic sources*
$\mathbf{E}$	$\mathbf{H}$
$\mathbf{H}$	$-\mathbf{E}$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
$\hat{y}$	$\hat{z}$
$\hat{z}$	$\hat{y}$
k	k
η	$1/\eta$

The concept of duality is important for several reasons. It is an aid to remembering equations, since almost half of them are duals of other equations. It shows us how to take the solution to one type of problem, interchange symbols, and obtain the solution to another type of problem. We can also use a physical or intuitive picture that applies to one type of problem and carry it over to the dual problem. For example, the picture of electric charge in motion giving rise to an electric current can also be used for magnetic case. That is, we can picture magnetic charge in motion as giving rise to magnetic current. Such a picture can serve as a guide to the mathematical development but cannot, of course, serve to argue for the existence of magnetic charges in nature. The concept of duality is based wholly on the mathematical symmetry of equations.

It is often convenient to divide a single problem into dual parts, thus cutting the mathematical labor in half. For example, suppose we have both electric and magnetic sources in a homogeneous medium of infinite extent. The field equations, Eqs. (3-1), are linear; so the total field can be considered as the sum of two parts, one produced by **J** and the other by **M**. To be explicit, let

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}'' \qquad \mathbf{H} = \mathbf{H}' + \mathbf{H}''$$

where $\quad \nabla \times \mathbf{H}' = \hat{y}\mathbf{E}' + \mathbf{J} \qquad -\nabla \times \mathbf{E}' = \hat{z}\mathbf{H}'$

and $\quad \nabla \times \mathbf{H}'' = \hat{y}\mathbf{E}'' \qquad -\nabla \times \mathbf{E}'' = \hat{z}\mathbf{H}'' + \mathbf{M}$

We have the solution for each of these partial problems in Table 3-1. The complete solution is therefore just the superposition of the two partial solutions, or

$$\begin{aligned} \mathbf{E} &= -\nabla \times \mathbf{F} + \hat{y}^{-1}(\nabla \times \nabla \times \mathbf{A} - \mathbf{J}) \\ \mathbf{H} &= \nabla \times \mathbf{A} + \hat{z}^{-1}(\nabla \times \nabla \times \mathbf{F} - \mathbf{M}) \end{aligned} \tag{3-4}$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$
$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{\mathbf{M}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \qquad (3\text{-}5)$$

We thus have the formal solution for any problem consisting of electric and magnetic currents in an unbounded homogeneous region. The above formulas are meant to include by implication sheets and filaments of currents.

It is instructive to show that *an infinitesimal dipole of magnetic current is indistinguishable from an infinitesimal loop of electric current.* We might suspect this from the circuit source representations of Fig. 3-1. However, rather than rely on this argument, let us consider the fields explicitly. A z-directed magnetic current dipole of moment Kl at the coordinate origin is the dual problem to the electric current dipole (Fig. 2-21). An interchange of symbols, according to Table 3-2, in Eqs. (2-113) will give us the field of the magnetic current element. For example, the electric intensity is

$$E_\phi = \frac{-Kl}{4\pi}\,e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)\sin\theta$$

The small loop of electric current is considered in Probs. 2-41 and 2-42 and is pictured in Fig. 2-26. Abstracting from Prob. 2-42, we have the electric intensity given by

$$E_\phi = \frac{\eta IS}{4\pi}\,e^{-jkr}\left(\frac{k^2}{r} - \frac{jk}{r^2}\right)\sin\theta$$

A comparison of the above two equations shows that they are identical if

$$Kl = j\omega\mu IS \qquad (3\text{-}6)$$

This equality is illustrated by Fig. 3-3. Thus, effect of an element of magnetic current can be realized in practice by a loop of electric current.

3-3. Uniqueness. A solution is said to be unique when it is the only one possible among a given class of solutions. It is important to have

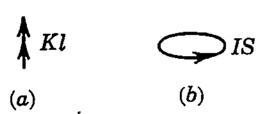

FIG. 3-3. These two sources radiate the same field if $Kl = j\omega\mu IS$. (a) Magnetic current element; (b) electric current loop.

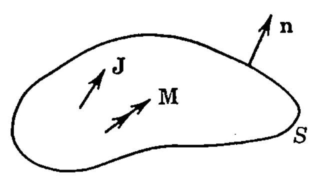

FIG. 3-4. S encloses linear matter and sources **J**, **M**.

precise theorems on uniqueness for several reasons. First of all, they tell us what information is needed to obtain the solution. Secondly, it is comforting to know that a solution is the only solution. Finally, uniqueness theorems establish conditions for a one-to-one correspondence of a field to its sources. This allows us to calculate the sources from a field, as well as the more usual reverse procedure.

Suppose we have a set of sources **J** and **M** acting in a region of linear matter bounded by the surface S, as suggested by Fig. 3-4. Any field within S must satisfy the complex field equations, Eqs. (3-1). Consider two possible solutions, $\mathbf{E}^a$, $\mathbf{H}^a$ and $\mathbf{E}^b$, $\mathbf{H}^b$. (These can be thought of as the fields when the sources *outside* of S are different.) We form the difference field $\delta\mathbf{E}$, $\delta\mathbf{H}$ according to

$$\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b \qquad \delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$$

Subtracting Eqs. (3-1) for the a field from those for the b field, we obtain

$$\left.\begin{aligned} -\nabla\times\delta\mathbf{E} &= \hat{z}\,\delta\mathbf{H} \\ \nabla\times\delta\mathbf{H} &= \hat{y}\,\delta\mathbf{E} \end{aligned}\right\} \quad \text{within } S$$

Thus, the difference field satisfies the source-free field equations within S. The conditions for uniqueness are those for which $\delta\mathbf{E} = \delta\mathbf{H} = 0$ everywhere within S, for then $\mathbf{E}^a = \mathbf{E}^b$ and $\mathbf{H}^a = \mathbf{H}^b$.

We now apply Eq. (1-54) to the difference field and obtain

$$\oiint (\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot d\mathbf{s} + \iiint (\hat{z}|\delta H|^2 + \hat{y}^*|\delta E|^2)\,d\tau = 0$$

Whenever

$$\oiint (\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot d\mathbf{s} = 0 \qquad (3\text{-}7)$$

over S, the volume integral must also vanish. Thus, if Eq. (3-7) is true, then

$$\iiint [\mathrm{Re}\,(\hat{z})|\delta H|^2 + \mathrm{Re}\,(\hat{y})|\delta E|^2]\,d\tau = 0$$
$$\iiint [\mathrm{Im}\,(\hat{z})|\delta H|^2 - \mathrm{Im}\,(\hat{y})|\delta E|^2]\,d\tau = 0 \qquad (3\text{-}8)$$

For dissipative media, Re $(\hat{z})$ and Re $(\hat{y})$ are always positive. If we assume some dissipation everywhere, however slight, then Eqs. (3-8) are satisfied only if $\delta\mathbf{E} = \delta\mathbf{H} = 0$ everywhere within S.

Some of the more important cases for which Eq. (3-7) is satisfied, and therefore uniqueness is obtained in lossy regions, are as follows. (1) The field is unique among a class **E**, **H** having $\mathbf{n}\times\mathbf{E}$ specified on S, for then $\mathbf{n}\times\delta\mathbf{E} = 0$ over S. (2) The field is unique among a class **E**, **H** having $\mathbf{n}\times\mathbf{H}$ specified on S, for then $\mathbf{n}\times\delta\mathbf{H} = 0$ over S. (3) The field is unique among a class **E**, **H** having $\mathbf{n}\times\mathbf{E}$ specified over part of S and $\mathbf{n}\times\mathbf{H}$ specified over the rest of S. These possibilities can be summarized by the following uniqueness theorem. *A field in a lossy region is uniquely* *specified by the sources within the region plus the tangential components of* **E** *over the boundary, or the tangential components of* **H** *over the boundary, or the former over part of the boundary and the latter over the rest of the boundary.* Note that our uniqueness proof breaks down for dissipationless media. To obtain uniqueness in this case, *we consider the field in a dissipationless medium to be the limit of the corresponding field in a lossy medium as the dissipation goes to zero.*

We have explicitly considered only volume distributions of sources and closed surfaces in our development, but the results are much more general than this. Singular sources, such as current sheets and current filaments, can be thought of as limiting cases of volume distributions and therefore are included by implication. Surfaces of infinite extent can be thought of as closed at infinity and can be included by appropriate limiting procedures. Of particular importance is the case for which the bounding surface is a sphere of radius $r \to \infty$, so that all space is included. If the sources are of finite extent, the vector potential solution of Eqs. (3-4) and (3-5) vanishes exponentially as $e^{-k''r}$, $r \to \infty$. We therefore have

$$\lim_{r\to\infty} \oint \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s} = 0 \tag{3-9}$$

for this solution (in lossy media). According to our uniqueness proof this must be the only solution for a class **E**, **H** satisfying Eq. (3-9). Thus, *given sources of finite extent in an unbounded lossy region, any solution satisfying Eq.* (3-9) *must be identically equal to the potential integral solution.* The loss-free case can be treated as the limit of the lossy case as dissipation vanishes.

To illustrate the above concepts, consider the current element of Fig. 2-21. Our solution at large r is Eq. (2-114). Let this be the a solution of our uniqueness proof, or

$$H_\phi{}^a = \frac{jIl}{2\lambda r} e^{-jkr} \sin\theta \qquad E_\theta{}^a = \eta H_\phi{}^a$$

It can be shown that the inward-traveling wave

$$H_\phi{}^b = \frac{-jIl}{2\lambda r} e^{jkr} \sin\theta \qquad E_\theta{}^b = -\eta H_\phi{}^b$$

is also a solution to the equations at large r. In Sec. 2-9, we threw out this second solution by reasoning that waves must travel outward from the source, not inward. Let us now consider these two solutions in the light of the uniqueness theorem. The difference field in this case is

$$\delta H_\phi = H_\phi{}^a - H_\phi{}^b = j\frac{Il}{\lambda r}\cos kr \sin\theta$$

$$\delta E_\theta = E_\theta{}^a - E_\theta{}^b = \eta\frac{Il}{\lambda r}\sin kr \sin\theta$$

In dissipationless media (k real), we can pick a sphere r = constant such that either δH_ϕ or δE_θ vanishes. Thus, Eq. (3-7) can be satisfied without obtaining uniqueness of the solution. However, in lossy media, sin kr and cos kr have no zeros $r > 0$, and Eq. (3-7) cannot be satisfied for any r. In this case, only the a solution vanishes as $r \to \infty$. It is therefore the desired solution in loss-free media.

3-4. Image Theory. Problems for which the field in a given region of space is determined from a knowledge of the field over the boundary of the region are called *boundary-value problems.* The rectangular waveguide of Sec. 2-7 is an example of a boundary-value problem. We shall now consider a class of boundary-value problems for which the boundary surface is a perfectly conducting plane. The procedure is known as image theory.

The boundary conditions at a perfect electric conductor are vanishing tangential components of **E**. An element of source plus an "image" element of source, radiating in free space, produce zero tangential components of **E** over the plane bisecting the line joining the two elements. According to uniqueness concepts, the solution to this problem is also the solution for a current element adjacent to a plane conductor. The necessary orientation and excitation of image elements is summarized by Fig. 3-5. Matter also can be imaged. For example, if a conducting sphere is adjacent to the plane conductor in the original problem, then two conducting spheres at image points are necessary in the image problem. In other words, we must maintain symmetry in the image problem. The procedure also applies to magnetic conductors in a dual sense. The application of image theory in a-c fields is much more restricted than in d-c fields. It is exact only when the plane conductor is perfect.

As an example of image theory, consider a current element normal to the ground (conducting) plane, as shown in Fig. 3-6a. This must produce the same field above the ground plane as do the two elements of Fig. 3-6b. Let us determine the radiation field. The radius vector from each current element is then parallel to that from the origin and given by

$$\left.\begin{aligned} r_o &= r - d\cos\theta \\ r_i &= r + d\cos\theta \end{aligned}\right\} \qquad r \gg d$$

where subscripts o and i refer to original and image elements, respectively. The radiation field of a single element is given by Eq. (2-114); so the radiation field of the two elements of Fig. 3-6b is the superposition

$$H_\phi = \frac{jIl}{2\lambda}\left(\frac{e^{-jkr_o}}{r_o} + \frac{e^{-jkr_i}}{r_i}\right)\sin\theta$$

$$\approx \frac{jIl}{\lambda r} e^{-jkr}\cos(kd\cos\theta)\sin\theta \tag{3-10}$$

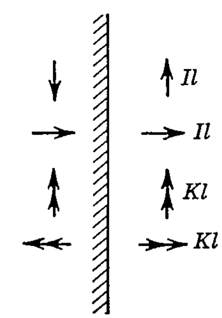

FIG. 3-5. A summary of image theory.

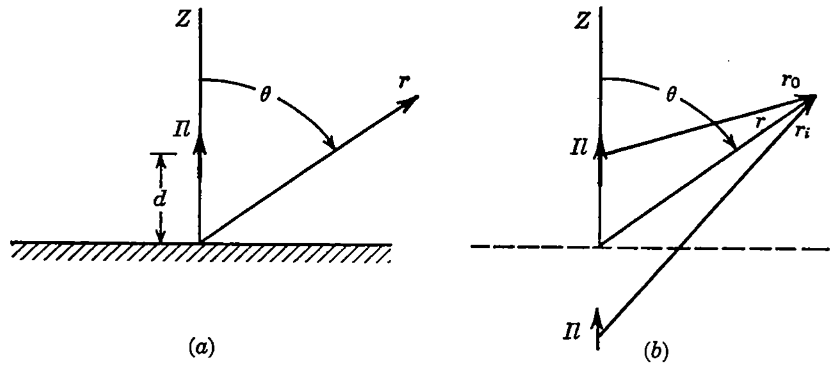

FIG. 3-6. A current element adjacent to a ground plane. (a) Original problem; (b) image problem.

and $E_\theta = \eta H_\phi$. According to image theory, this must also be the solution to Fig. 3-6a above the ground plane.

The problem of Fig. 3-6a represents the antenna system of a short dipole antenna adjacent to a ground plane. The total power radiated by the system is

$$\bar{\mathcal{P}}_f = \iint_{\substack{\text{hemi-}\\\text{sphere}}} E_\theta H_\phi^* \, ds = 2\pi\eta \int_0^{\pi/2} |H_\phi|^2 r^2 \sin\theta \, d\theta$$

where integration is over the large hemisphere $z > 0$, $r \to \infty$. Substituting from Eq. (3-10) and integrating, we have

$$\bar{\mathcal{P}}_f = 2\pi\eta \left| \frac{Il}{\lambda} \right|^2 \left[\frac{1}{3} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3} \right] \tag{3-11}$$

As $kd \to \infty$, the power radiated is equal to that radiated by an isolated element [Eq. (2-116)]. As $kd \to 0$, the power radiated is double that radiated by an isolated element. The gain of the antenna system over an omnidirectional radiator, according to Eq. (2-130), is

$$g = \frac{4\pi r^2 \eta |H_\phi|^2}{\bar{\mathcal{P}}_f} = \frac{2}{\dfrac{1}{3} - \dfrac{\cos 2kd}{(2kd)^2} + \dfrac{\sin 2kd}{(2kd)^3}} \tag{3-12}$$

along the ground plane. This is $g = 3$ at $kd = 0$, and $g = 6$ as $kd \to \infty$. The maximum gain occurs at $kd = 2.88$, for which $g = 6.57$. Thus, a gain of more than four times that of the isolated element (1.5) can be achieved. Figure 3-7 shows the radiation field patterns for the cases

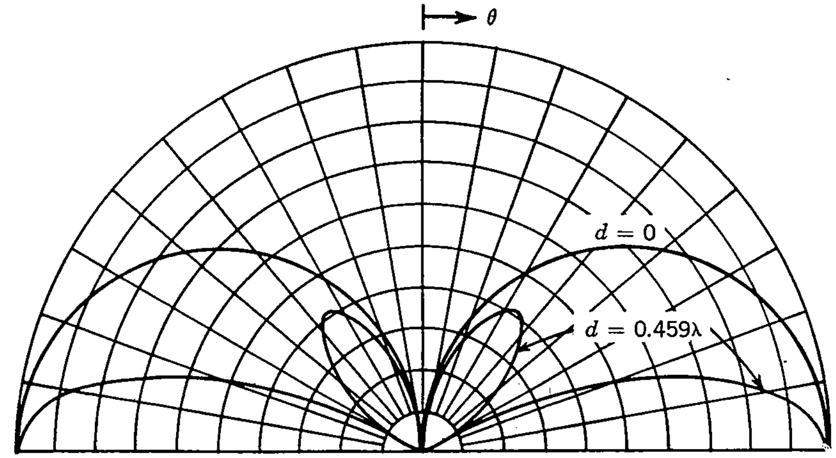

FIG. 3-7. Radiation field patterns for the current element of Fig. 3-6a.

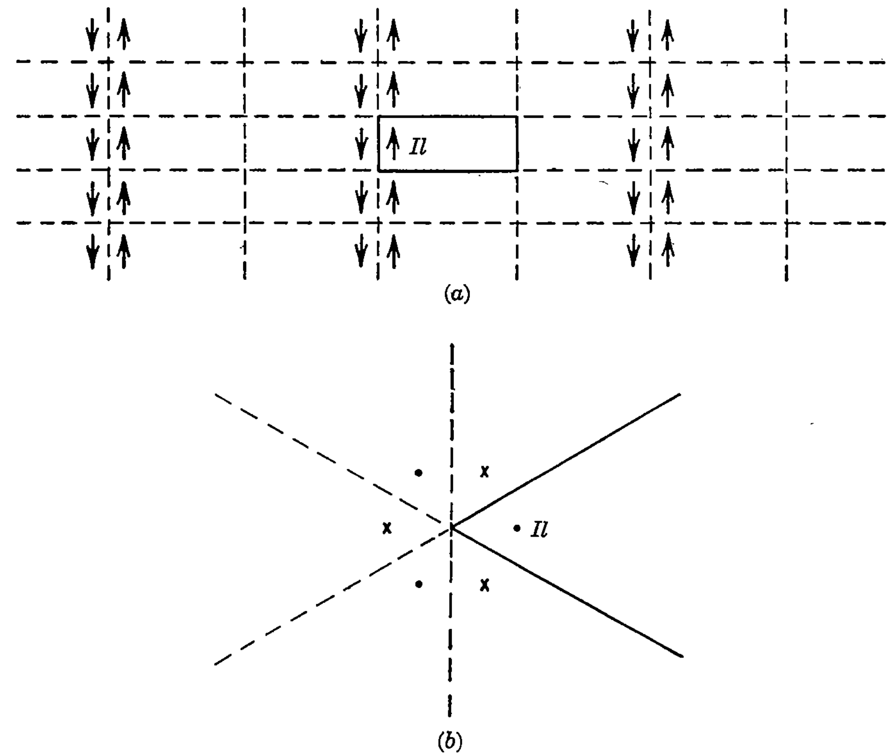

FIG. 3-8. Problems involving multiple images. (a) Current element in a conducting tube; (b) current element in a conducting wedge.

$d = 0$ (element at the gound plane surface) and $d = 0.459\lambda$ (maximum gain).

Image theory also can be applied in certain problems involving more than one conducting plane. Two such cases are illustrated by Fig. 3-8. In the case of a conducting tube (Fig. 3-8*a*), an infinite lattice of images is needed. In the case of a conducting wedge (Fig. 3-8*b*), a finite set of images results. Image theory can be used for conducting wedges when the wedge angle is $180°/n$ (n an integer).

3-5. The Equivalence Principle. Many source distributions outside a given region can produce the same field inside the region. For example, the image current element of Fig. 3-6*b* produces the same field above the plane $z = 0$ as do the currents on the conductor of Fig. 3-6*a*. Two sources producing the same field within a region of space are said to be *equivalent within that region.* When we are interested in the field in a given region of space, we do not need to know the actual sources. Equivalent sources will serve as well.

A simple application of the equivalence principle is illustrated by Fig. 3-9. Let Fig. 3-9*a* represent a source (perhaps a transmitter and antenna) internal to S and free space external to S. We can set up a problem equivalent to the original problem external to S as follows. Let the original field exist external to S, and the null field internal to S, with free space everywhere. This is shown in Fig. 3-9*b*. To support this field, there must exist surface currents $\mathbf{J}_s$, $\mathbf{M}_s$ on S according to Eqs. (1-86). These currents are therefore

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H} \qquad \mathbf{M}_s = \mathbf{E} \times \mathbf{n} \tag{3-13}$$

where $\mathbf{n}$ points outward and $\mathbf{E}$, $\mathbf{H}$ are the original fields over S. Since the currents act in unbounded free space, we can determine the field from them by Eqs. (3-4) and (3-5). From the uniqueness theorem, we know that the field so calculated will be the originally postulated field, that is, $\mathbf{E}$, $\mathbf{H}$ external to S and zero internal to S. The final result of this procedure is a formula for $\mathbf{E}$ and $\mathbf{H}$ everywhere external to S in terms of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ on S.

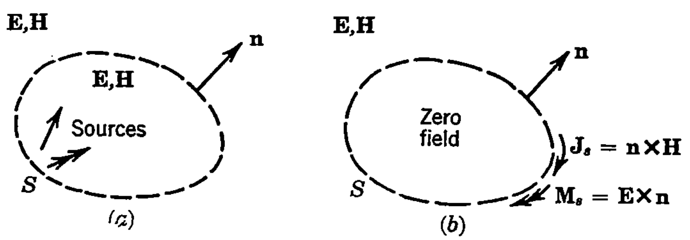

Fig. 3-9. The equivalent currents produce the same field external to S as do the original sources.

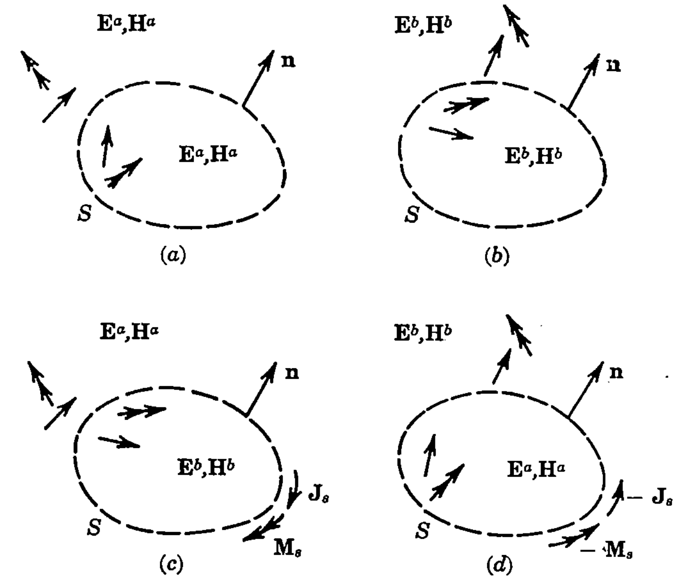

Fig. 3-10. A general formulation of the equivalence principle. (*a*) Original *a* problem; (*b*) original *b* problem; (*c*) equivalent to *a* external to S and to *b* internal to S; (*d*) equivalent to *b* external to S and to *a* internal to S.

We were overly restrictive in specifying the null field internal to S in the preceding example. Any other field would serve as well, giving us infinitely many equivalent currents as far as the external region is concerned. This general formulation of the equivalence principle is represented by Fig. 3-10. We have two original problems consisting of currents in linear media, as shown in Fig. 3-10*a* and *b*. We can set up a problem equivalent to *a* external to S and equivalent to *b* internal to S as follows. External to S, we specify that the field, medium, and sources remain the same as in the *a* problem. Internal to S, we specify that the field, medium, and sources remain the same as in the *b* problem. To support this field, there must be surface currents $\mathbf{J}_s$ and $\mathbf{M}_s$ on S. According to Eqs. (1-86), these are given by

$$\mathbf{J}_s = \mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) \qquad \mathbf{M}_s = (\mathbf{E}^a - \mathbf{E}^b) \times \mathbf{n} \tag{3-14}$$

where $\mathbf{E}^a$, $\mathbf{H}^a$ is the field of the *a* problem and $\mathbf{E}^b$, $\mathbf{H}^b$ is the field of the *b* problem. This equivalent problem is shown in Fig. 3-10*c*. We can also set up a problem equivalent to *b* external to S and to *a* internal to S in an analogous manner, as shown in Fig. 3-10*d*. In this case the necessary surface currents are the negative of Eqs. (3-14). Note that in each case we must keep the original sources and media in the region for which we keep the field. Note also that we cannot use Eqs. (3-4) and (3-5) to

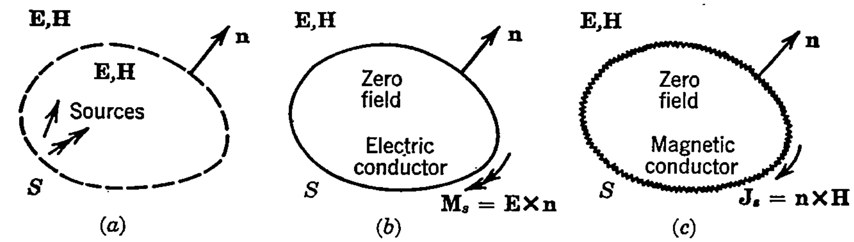

FIG. 3-11. The field external to S is the same in (a), (b), and (c). (a) Original problem; (b) magnetic current backed by an electric conductor; (c) electric current backed by a magnetic conductor.

determine the field of the currents unless the equivalent currents radiate into an unbounded homogeneous region. Finally, note that the restricted form of the equivalence principle (Fig. 3-9) is the special case of the general form for which all a sources and matter lie inside S and all b sources are zero.

So far, we have used the tangential components of *both* **E** and **H** in setting up our equivalent problems. From uniqueness concepts, we know that the tangential components of only **E** *or* **H** are needed to determine the field. We shall now show that equivalent problems can be found in terms of only magnetic currents (tangential **E**) or only electric currents (tangential **H**).

Consider a problem for which all sources lie within S, as shown in Fig. 3-11a. We set up the equivalent problem of Fig. 3-11b as follows. Over S we place a perfect electric conductor, and on top of this we place a sheet of magnetic current $\mathbf{M}_s$. External to S we specify the same field and medium as in the original problem. Since the tangential components of **E** are zero on the conductor (just behind $\mathbf{M}_s$), and equal to the original field components just in front of $\mathbf{M}_s$, it follows from Eqs. (1-86) that

$$\mathbf{M}_s = \mathbf{E} \times \mathbf{n} \tag{3-15}$$

We now have the same tangential components of **E** over S in both Fig. 3-11a and b; so according to our uniqueness theorem the field outside of S must be the same in both cases. We can derive the alternative equivalent problem of Fig. 3-11c in an analogous manner. For this we need the perfect magnetic conductor, that is, a boundary of zero tangential components of **H**. We then find that the electric current sheet

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H} \tag{3-16}$$

over a perfect magnetic conductor covering S produces the same field external to S as do the original sources.

By now, the general philosophy of the equivalence principle should be apparent. It is based upon the one-to-one correspondence between fields and sources when uniqueness conditions are met. If we specify the field and matter everywhere in space, we can determine all sources. We derived our various equivalences in this manner.

Considerable physical interpretation can be given to the equivalence principle. For example, in the problem of Fig. 3-9b, the field internal to S is zero. It therefore makes no difference what matter is within S as far as the field external to S is concerned. We have previously assumed that free space existed within S, so that the potential integral solution could be applied. We could just as well introduce a perfect electric conductor to back the current sheets of Fig. 3-9b. It can be shown by reciprocity (Sec. 3-8) that an electric current just in front of an electric current conductor produces no field. (We can think of the conductor as shorting out the current.) Therefore, the field is produced by the magnetic currents alone, in the presence of the electric conductor, which is Fig. 3-11b. Alternatively, we could back the equivalent currents of Fig. 3-9b with a perfect magnetic conductor and obtain the equivalent problem of Fig. 3-11c. When matter is placed within S in Fig. 3-9b, the partial fields produced by $\mathbf{J}_s$ alone and $\mathbf{M}_s$ alone will change external to S, but the total field must remain unchanged.

Perhaps it would help us to understand the equivalence principle if we considered the analogous concept in circuit theory. Consider a source (active network) connected to a passive network, as shown in Fig. 3-12a. We can set up a problem equivalent to this as far as the passive network is concerned, as follows. The original source is switched off, leaving the source impedance connected. A current source I, equal to the terminal current in the original problem, is placed across the terminals. A voltage

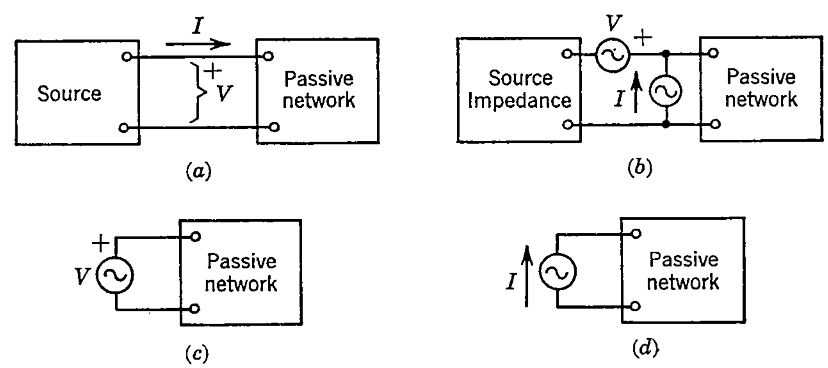

FIG. 3-12. A circuit theory analogue to the equivalence principle. (a) Original problem; (b) equivalent sources; (c) source impedance replaced by a short circuit; (d) source impedance replaced by an open circuit.

source V, equal to the terminal voltage in the original problem, is placed in series with the interconnection. This is illustrated by Fig. 3-12*b*. It is evident from the usual circuit concepts that there is no excitation of the source impedance from these equivalent sources, whereas the excitation of the passive network is unchanged. Thus, Fig. 3-12*b* is the circuit analogue to Fig. 3-9*b*.

Since there is no excitation of the source impedance in Fig. 3-12*b*, we may replace it by an arbitrary impedance without affecting the excitation of the passive network. This is analogous to the arbitrary placement of matter within S in the field equivalence of Fig. 3-9*b*. In particular, let the source impedance be replaced by a short circuit. This short-circuits the current source and leaves only the voltage source exciting the network (recall circuit theory superposition). Thus, the voltage source alone, as illustrated by Fig. 3-12*c*, produces the same excitation of the passive network as does the original source. This is analogous to the field problem of Fig. 3-11*b*. Now consider the source impedance of Fig. 3-12*b* replaced by an open circuit. This leaves only the current source exciting the network, as shown in Fig. 3-12*d*. This is analogous to the field problem of Fig. 3-11*c*.

3-6. Fields in Half-space. A combination of the equivalence principle and image theory can be used to obtain solutions to boundary-value problems for which the field in half-space is to be determined from its tangential components over the bounding plane. To illustrate, let the original problem consist of matter and sources $z < 0$, and free space $z > 0$, as shown in Fig. 3-13*a*. An application of the equivalence concepts of Fig. 3-11*b* yields the equivalent problem of Fig. 3-13*b*. This consists of the magnetic currents of Eq. (3-15) adjacent to an infinite

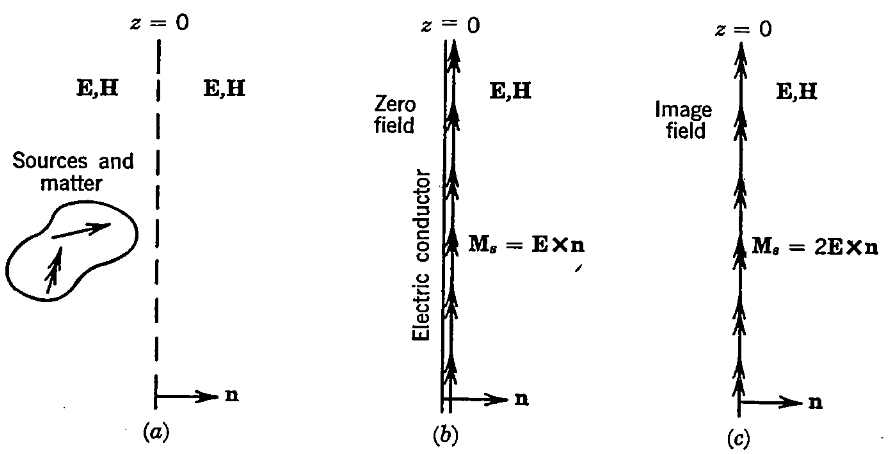

FIG. 3-13. Illustration of the steps used to establish Eq. (3-17).

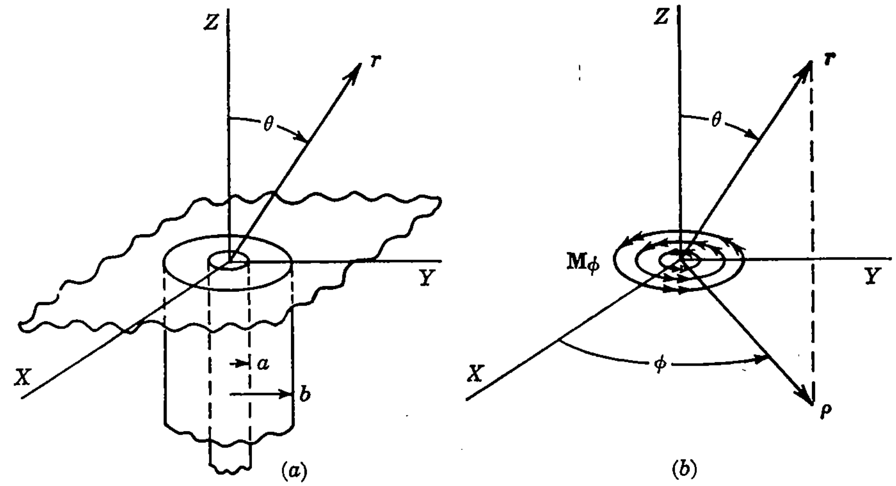

FIG. 3-14. A coaxial line opening onto a ground plane. (*a*) Original problem; (*b*) equivalent problem.

ground plane. We now image the magnetic currents in the ground plane, according to Fig. 3-5. The images are equal in magnitude to, and essentially coincident with, the $\mathbf{M}_s$ of Fig. 3-13*b*. Thus, as pictured in Fig. 3-13*c*, the magnetic currents $2\mathbf{M}_s$ radiating into unbounded space produce the same field $z > 0$ as do the original sources. They produce an image field $z < 0$, which is of no interest to us. The field of Fig. 3-13*c* is then calculated according to Eqs. (3-4) and (3-5) with $\mathbf{A} = 0$. This can be summarized mathematically by

$$\mathbf{E}(\mathbf{r}) = -\nabla \times \iint\limits_{\text{plane}} \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{E}(\mathbf{r}') \times d\mathbf{s}' \tag{3-17}$$

This is a mathematical identity valid for any field $\mathbf{E}$ satisfying Eq. (2-3). The $\mathbf{H}$ field satisfies Eq. (2-4), which is identical to Eq. (2-3); so the above identity must also be valid for $\mathbf{E}$ replaced by $\mathbf{H}$. We can show this by reasoning dual to that used to establish Eq. (3-17).

The above result is particularly useful for problems involving apertures in conducting ground planes. As an example, suppose we have a coaxial transmission line opening into a ground plane (Fig. 3-14*a*). According to the above discussion, the field must be the same as that produced by Fig. 3-14*b*. Note that $\mathbf{M}_s$ exists only over the aperture (coax opening), for tangential $\mathbf{E}$ is zero over the ground plane. Let us asume that the field over the aperture is the transmission-line mode of the coax, that is

$$E_\rho = \frac{-V}{\rho \log (b/a)}$$

where V is the line voltage. To this approximation, the magnetic current in Fig. 3-14b is

$$M_\phi = \frac{V}{\rho \log (b/a)}$$

This is a loop of magnetic current which, if $b \ll \lambda$, acts as an electric dipole (dual to Fig. 3-3). Visualize this current as a continuous distribution of magnetic current filaments of strength $dK = M_\phi \, d\rho$. The total moment of the source is then

$$KS = \int \pi\rho^2 \, dK = \frac{\pi V}{\log (b/a)} \int_a^b \rho \, d\rho = \frac{\pi V(b^2 - a^2)}{2 \log (b/a)} \tag{3-18}$$

The equivalent electric current element must satisfy the equation dual to Eq. (3-6), or

$$Il = -j\omega\epsilon KS \tag{3-19}$$

We have now reduced the problem to that of Fig. 3-6a with $kd = 0$. From Eq. (3-10) and the above equalities, we have the radiation field given by

$$H_\phi = \frac{\omega\epsilon\pi V(b^2 - a^2)}{2\lambda r \log (b/a)} e^{-jkr} \sin \theta \tag{3-20}$$

and $E_\theta = \eta H_\phi$. Thus, the radiation field pattern is the $d = 0$ curve of Fig. 3-7. The gain of the antenna system is $g = 3$.

The power radiated is Eq. (3-11) with $kd = 0$ and Il given by Eqs. (3-18) and (3-19), or

$$\bar{\mathcal{P}}_f = 2\pi\eta \left| \frac{\omega\epsilon\pi V(b^2 - a^2)}{2\lambda \log (b/a)} \right|^2 \frac{2}{3} = \frac{4\pi}{3\eta} \left| \frac{\pi^2(b^2 - a^2)V}{\lambda^2 \log (b/a)} \right|^2 \tag{3-21}$$

Note that the power radiated varies inversely as λ^4. Note also that our answers are referred to a voltage, characteristic of aperture antennas. This is in contrast to answers referred to current for wire antennas. For aperture antennas we define a *radiation conductance* according to

$$G_r = \frac{\bar{\mathcal{P}}_f}{|V|^2} \tag{3-22}$$

where V is an arbitrary reference voltage. In the coaxial radiator of Fig. 3-14 it is logical to pick this V to be the coaxial V at the aperture. Hence, the radiation conductance is

$$G_r = \frac{4\pi^5}{3\eta} \left[\frac{b^2 - a^2}{\lambda^2 \log (b/a)} \right]^2 \tag{3-23}$$

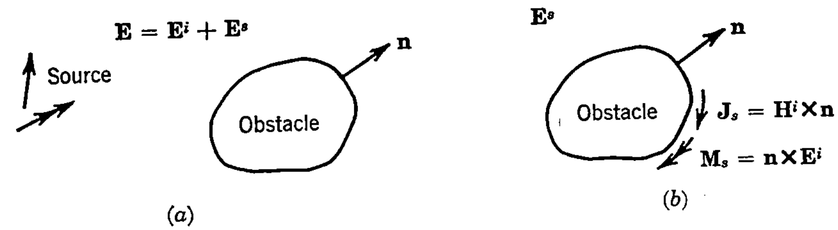

FIG. 3-15. Illustration of the induction theorem. (*a*) Original problem; (*b*) induction equivalent.

For the usual coaxial line, G_r is small, and the coaxial line sees nearly an open circuit. As a and b are made larger, the radiation becomes more pronounced, but our formulas must then be modified.[1]

3-7. The Induction Theorem. We now consider a theorem closely related in concept to the equivalence principle. Consider a problem in which a set of sources are radiating in the presence of an obstacle (material body). This is illustrated by Fig. 3-15a. Define the *incident field* $\mathbf{E}^i$, $\mathbf{H}^i$ as the field of the sources with the obstacle absent. Define the *scattered field* $\mathbf{E}^s$, $\mathbf{H}^s$ as the difference between the field with the obstacle present ($\mathbf{E}$, $\mathbf{H}$) and the incident field, that is,

$$\mathbf{E}^s = \mathbf{E} - \mathbf{E}^i \qquad \mathbf{H}^s = \mathbf{H} - \mathbf{H}^i \tag{3-24}$$

This scattered field can be thought of as the field produced by the currents (conduction and polarization) on the obstacle. External to the obstacle, both $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{E}^i$, $\mathbf{H}^i$ have the same sources. The scattered field $\mathbf{E}^s$, $\mathbf{H}^s$ is therefore a source-free field external to the obstacle.

We now construct a second problem as follows. Retain the obstacle, and postulate that the original field $\mathbf{E}$, $\mathbf{H}$ exists internal to it and that the scattered field $\mathbf{E}^s$, $\mathbf{H}^s$ exists external to it. Both these fields are source-free in their respective regions. To support these fields, there must be surface currents on S according to Eqs. (1-86), that is,

$$\mathbf{J}_s = \mathbf{n} \times (\mathbf{H}^s - \mathbf{H}) \qquad \mathbf{M}_s = (\mathbf{E}^s - \mathbf{E}) \times \mathbf{n}$$

where $\mathbf{n}$ points outward from S. According to Eqs. (3-24), these reduce to

$$\mathbf{J}_s = \mathbf{H}^i \times \mathbf{n} \qquad \mathbf{M}_s = \mathbf{n} \times \mathbf{E}^i \tag{3-25}$$

It follows from the uniqueness theorem that these currents, radiating in the presence of the obstacle, produce the postulated field ($\mathbf{E}$, $\mathbf{H}$ internal to S, and $\mathbf{E}^s$, $\mathbf{H}^s$ external to S). This is the *induction theorem*, illustrated by Fig. 3-15b.

It is instructive to compare the induction theorem with the equiva-

[1] H. Levine and C. H. Papas, Theory of the Circular Diffraction Antenna, *J. Appl. Phy.*, vol. 22, no. 1, pp. 29–43, January, 1951.

lence theorem. The latter postulates **E**, **H** internal to S and zero field external to S, which must be supported by currents

$$\mathbf{J}_s = \mathbf{H} \times \mathbf{n} \qquad \mathbf{M}_s = \mathbf{n} \times \mathbf{E}$$

on S. These currents can be considered as radiating into an unbounded medium having constitutive parameters equal to those of the obstacle. Thus, we can use Eqs. (3-4) and (3-5) to calculate the field of the above currents. However, we do not know $\mathbf{J}_s$ and $\mathbf{M}_s$ until we know **E**, **H** on S, that is, until we have the solution to the problem of Fig. 3-15*a*. We can, however, approximate $\mathbf{J}_s$ and $\mathbf{M}_s$ and from these calculate an approximation to **E**, **H** within S.

In contrast to the above, the induction theorem yields known currents [Eqs. (3-25)]. (This assumes that $\mathbf{E}^i$, $\mathbf{H}^i$ is known.) We cannot, however, use Eqs. (3-4) and (3-5) to calculate the field from $\mathbf{J}_s$, $\mathbf{M}_s$, for they radiate in the presence of the obstacle. A determination of this field is a boundary-value problem of the same order of complexity as the original problem (Fig. 3-15*a*). We can, however, approximate the field of $\mathbf{J}_s$, $\mathbf{M}_s$ and thereby obtain an approximate formula for **E**, **H** internal to S and $\mathbf{E}^s$, $\mathbf{H}^s$ external to S.

A simplification of the induction theorem occurs when the obstacle is a perfect conductor. This situation is represented by Fig. 3-16*a*. The solution **E** must satisfy the boundary condition $\mathbf{n} \times \mathbf{E} = 0$ on S (zero tangential **E**). It then follows from the first of Eqs. (3-24) that

$$\mathbf{n} \times \mathbf{E}^s = -\mathbf{n} \times \mathbf{E}^i \qquad \text{on } S \tag{3-26}$$

We now know the tangential components of $\mathbf{E}^s$ over S; so we can construct the induction representation of Fig. 3-16*b* as follows. We keep the perfectly conducting obstacle and specify that external to S the field $\mathbf{E}^s$, $\mathbf{H}^s$ exists. To support this field, there must be magnetic currents on S given by

$$\mathbf{M}_s = \mathbf{E}^s \times \mathbf{n} = \mathbf{n} \times \mathbf{E}^i \tag{3-27}$$

We can visualize this current as causing the tangential components of **E** to jump from zero at the conductor to those of $\mathbf{E}^s$ just outside $\mathbf{M}_s$. The

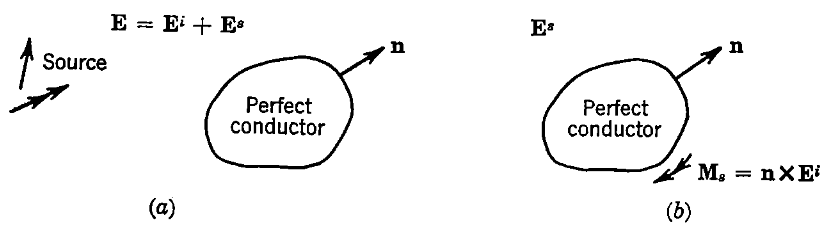

FIG. 3-16. The induction theorem as applied to a perfectly conducting obstacle. (*a*) Original problem; (*b*) induction equivalent.

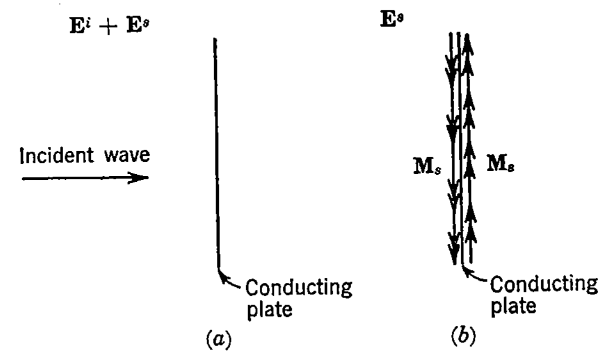

FIG. 3-17. Scattering by a conducting plate. (*a*) Original problem; (*b*) induction equivalent.

tangential components of **E** in Fig. 3-16*b* therefore have been forced to be $\mathbf{E}^s$. Thus, according to uniqueness concepts, the currents of Eq. (3-27) radiating in the presence of the conducting obstacle must produce $\mathbf{E}^s$, $\mathbf{H}^s$ external to S.

It is interesting to compare this result with the previous one (Fig. 3-15*b*). We found that, in general, both electric and magnetic currents exist on S in the induction representation. How, then, can both Fig. 3-15*b* and Fig. 3-16*b* be correct for a perfectly conducting obstacle? The answer must be that an electric current impressed along a perfect electric conductor produces no field. If the conductor is plane, this is evident from image theory. We can prove it, in general, by using the reciprocity concepts of the next section.

To illustrate an application of the induction theorem, consider the problem of determining the back scattering, or radar echo, from a large conducting plate. This problem is suggested by Fig. 3-17*a*. For normal incidence, let the plate lie in the $z = 0$ plane and let the incident field be specified by

$$E_x^i = E_0 e^{-jkz} \tag{3-28}$$

According to the induction theorem, the scattered field is produced by the currents $M_y = E_0$ on the side facing the source and $M_y = -E_0$ on the side away from the source. These currents radiate in the presence of the original conducting plate, as represented by Fig. 3-17*b*. Let the field from each element of current be approximated by the field from an element adjacent to a ground plane. According to image theory, this means that each element of M_y seen by the receiver radiates as $2M_y = 2E_0$ in free space. Hence, far from the plate, it contributes

$$dE_x^s = \frac{-jkE_0\, ds}{2\pi r} e^{-jkr}$$

in the back-scatter direction. Each element not seen by the receiver contributes nothing to the back-scattered field. Summing over the entire plate, we have the distant back-scattered field given by

$$E_x^s = \iint_{\text{plate}} dE_x^s = \frac{-jkE_0A}{2\pi r} e^{-jkr} \tag{3-29}$$

where A is the area of the plate.

The *echo area* or *radar cross section* of an obstacle is defined as the area for which the incident wave contains sufficient power to produce, by omnidirectional radiation, the same back-scattered power density. In mathematical form, the echo area is

$$A_e = \lim_{r \to \infty} \left(4\pi r^2 \frac{\mathcal{S}^s}{\mathcal{S}^i} \right) \tag{3-30}$$

where $\mathcal{S}^i$ is the incident power density and $\mathcal{S}^s$ is the scattered power density. For our problem, $\mathcal{S}^i = |E_0|^2/\eta$ and, from Eq. (3-29),

$$\mathcal{S}^s = \frac{1}{\eta} \left| \frac{kE_0A}{2\pi r} \right|^2$$

The echo area of a conducting plate is therefore

$$A_e \approx \frac{k^2A^2}{\pi} = \frac{4\pi A^2}{\lambda^2} \tag{3-31}$$

valid for large plates and normal incidence.

3-8. Reciprocity. In its simplest sense, a reciprocity theorem states that a response of a system to a source is unchanged when source and measurer are interchanged. In a more general sense, reciprocity theorems relate a response at one source due to a second source to the response at the second source due to the first source. We shall establish this type of reciprocity relationship for a-c fields. The reciprocity theorem of circuit theory is a special case of this reciprocity theorem for fields.

Consider two sets of a-c sources, $\mathbf{J}^a$, $\mathbf{M}^a$ and $\mathbf{J}^b$, $\mathbf{M}^b$, of the same frequency, existing in the same linear medium. Denote the field produced by the a sources alone by $\mathbf{E}^a$, $\mathbf{H}^a$, and the field produced by the b sources alone by $\mathbf{E}^b$, $\mathbf{H}^b$. The field equations are then

$$\begin{aligned} \nabla \times \mathbf{H}^a &= \hat{y}\mathbf{E}^a + \mathbf{J}^a & \qquad \nabla \times \mathbf{H}^b &= \hat{y}\mathbf{E}^b + \mathbf{J}^b \\ -\nabla \times \mathbf{E}^a &= \hat{z}\mathbf{H}^a + \mathbf{M}^a & \qquad -\nabla \times \mathbf{E}^b &= \hat{z}\mathbf{H}^b + \mathbf{M}^b \end{aligned}$$

We multiply the first equation scalarly by $\mathbf{E}^b$ and the last equation by $\mathbf{H}^a$ and add the resulting equations. This gives

$$-\nabla \cdot (\mathbf{E}^b \times \mathbf{H}^a) = \hat{y}\mathbf{E}^a \cdot \mathbf{E}^b + \hat{z}\mathbf{H}^a \cdot \mathbf{H}^b + \mathbf{E}^b \cdot \mathbf{J}^a + \mathbf{H}^a \cdot \mathbf{M}^b$$

where the left-hand term has been simplified by the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

An interchange of a and b in this result gives

$$-\nabla \cdot (\mathbf{E}^a \times \mathbf{H}^b) = \hat{y}\mathbf{E}^a \cdot \mathbf{E}^b + \hat{z}\mathbf{H}^a \cdot \mathbf{H}^b + \mathbf{E}^a \cdot \mathbf{J}^b + \mathbf{H}^b \cdot \mathbf{M}^a$$

A subtraction of the former equation from the latter yields

$$-\nabla \cdot (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) = \mathbf{E}^a \cdot \mathbf{J}^b + \mathbf{H}^b \cdot \mathbf{M}^a - \mathbf{E}^b \cdot \mathbf{J}^a - \mathbf{H}^a \cdot \mathbf{M}^b \tag{3-32}$$

At any point for which the fields are source-free ($\mathbf{J} = \mathbf{M} = 0$), this reduces to

$$\nabla \cdot (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) = 0 \tag{3-33}$$

which is called the *Lorentz reciprocity theorem.* If Eq. (3-33) is integrated throughout a source-free region and the divergence theorem applied, we have

$$\oint\!\!\!\!\oint (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot d\mathbf{s} = 0 \tag{3-34}$$

which is the integral form of the Lorentz reciprocity theorem for a source-free region.

For a region containing sources, integration of Eq. (3-32) throughout the region gives

$$-\oint\!\!\!\!\oint (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot d\mathbf{s} = \iiint (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{H}^a \cdot \mathbf{M}^b - \mathbf{E}^b \cdot \mathbf{J}^a + \mathbf{H}^b \cdot \mathbf{M}^a)\, d\tau \tag{3-35}$$

Let us now postulate that all sources and matter are of finite extent. Distant from the sources and matter, we have (see Sec. 3-13)

$$E_\theta = \eta H_\phi \qquad E_\phi = -\eta H_\theta$$

The left-hand term of Eq. (3-35), integrated over a sphere of radius $r \to \infty$, is then

$$-\eta \oint\!\!\!\!\oint (H_\theta{}^a H_\theta{}^b + H_\phi{}^a H_\phi{}^b - H_\theta{}^b H_\theta{}^a - H_\phi{}^b H_\phi{}^a)\, ds = 0$$

Equation (3-35) now reduces to

$$\iiint (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{H}^a \cdot \mathbf{M}^b)\, d\tau = \iiint (\mathbf{E}^b \cdot \mathbf{J}^a - \mathbf{H}^b \cdot \mathbf{M}^a)\, d\tau \tag{3-36}$$

where the integration extends over all space. This is the most useful form of the reciprocity theorem for our purposes. Equation (3-36) also applies to regions of finite extent whenever Eq. (3-34) is satisfied. For example, fields in a region bounded by a perfect electric conductor satisfy Eq. (3-34); hence Eq. (3-36) applies in this case.

The integrals appearing in Eq. (3-36) *do not* in general represent power, since no conjugates appear. They have been given the name *reaction.*[1] By definition, the reaction of field a on source b is

$$\langle a,b\rangle = \iiint (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{H}^a \cdot \mathbf{M}^b)\, d\tau \tag{3-37}$$

In this notation, the reciprocity theorem is

$$\langle a,b\rangle = \langle b,a\rangle \tag{3-38}$$

that is, the reaction of field a on source b is equal to the reaction of field b on source a. Reaction is a useful quantity primarily because of this conservative property. For example, reaction can be used as a measure of equivalency, since a source must have the same reaction with all fields equivalent over its extent. This equality of reaction is a necessary, but not sufficient, test of equivalence as defined in Sec. 3-5. We shall use the term *self-reaction* to denote the reaction of a field on its own sources, that is, $\langle a,a\rangle$.

A valuable tool for expositional purposes can be obtained by using the circuit sources of Fig. 3-1 in the reaction concept. For a current source (Fig. 3-1a), we have

$$\langle a,b\rangle = \int \mathbf{E}^a \cdot I^b\, d\mathbf{l} = I^b \int \mathbf{E}^a \cdot d\mathbf{l} = -V^a I^b$$

where V^a is the voltage across the b source due to some (as yet unspecified) a source. For a voltage source (Fig. 3-1b), we have $K^b = -V^b$, and

$$\langle a,b\rangle = -\oint \mathbf{H}^a \cdot K^b\, d\mathbf{l} = -K^b \oint \mathbf{H}^a \cdot d\mathbf{l} = V^b I^a$$

where I^a is the current through the b source due to some a source. To summarize, the "circuit reactions" are

$$\langle a,b\rangle = \begin{cases} -V^a I^b & b \text{ a current source} \\ +V^b I^a & b \text{ a voltage source} \end{cases} \tag{3-39}$$

If we use a unit current source ($I^b = 1$), then $\langle a,b\rangle$ is a measure of V^a (the voltage at b due to another source a). If we use a unit voltage source ($V^b = 1$), then $\langle a,b\rangle$ is a measure of I^a (the current at b due to another source a).

To relate our reciprocity theorem to the usual circuit theory statement of reciprocity, consider the two-port (four-terminal) network of

[1] V. H. Rumsey, The Reaction Concept in Electromagnetic Theory, *Phys. Rev.*, ser. 2, vol. 94, no. 6, pp. 1483–1491, June 15, 1954.

Fig. 3-18. The characteristics of a linear network can be described by the impedance matrix $[z]$ defined by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3-40}$$

Suppose we apply a current source I_1 at port 1 and a current source I_2 at port 2. Let the partial response V_{ij} be the voltage at port i due to source I_j at port j. Each current source sees the other port open-circuited (see Fig. 3-1a); hence

$$z_{ij} = \frac{V_{ij}}{I_j}$$

In terms of the circuit reactions [Eq. (3-39)], $\langle j,i\rangle = -V_{ij}I_i$; hence

$$z_{ij} = -\frac{\langle j,i\rangle}{I_i I_j} \tag{3-41}$$

Thus, the elements of the impedance matrix are the various reactions among two unit current sources. The reciprocity theorem [Eq. (3-38)], applied to Eq. (3-41), shows that

$$z_{ij} = z_{ji} \tag{3-42}$$

which is the usual statement of reciprocity in circuit theory. Equations (3-41) and (3-42) also apply to an N-port network. The use of voltage sources instead of current sources gives reactions proportional to the elements of the admittance matrix $[y]$, and reciprocity then states that $y_{ij} = y_{ji}$.

The proofs of many other theorems can be based on the reciprocity theorem. For example, the preceding paragraph is a proof that *any network constructed of linear isotropic matter has a symmetrical impedance matrix.* This "network" might be the two antennas of Fig. 3-19. Reciprocity in this case can be stated as: The voltage at b due to a current source at a is equal to the voltage at a due to the same current source at b. If the b antenna is infinitely remote from the a antenna, its field will be a plane wave in the vicinity of a, and vice versa. The *receiving pattern* of an antenna is defined as the voltage at the antenna

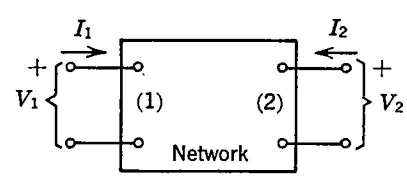

FIG. 3-18. A two-port network.

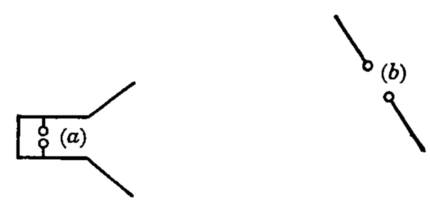

FIG. 3-19. Two antennas.

terminals due to a plane wave incident upon the antenna. The reciprocity theorem for antennas can thus be stated as: *The receiving pattern of any antenna constructed of linear isotropic matter is identical to its transmitting pattern.*

In Secs. 3-5 and 3-7, we used the fact that an electric current impressed along the surface of a perfect electric conductor radiated no field. The reciprocity theorem proves this, in general, as follows. Visualize a set of terminals a on the conductor and another set of terminals b in space away from the conductor. A current element at b produces no tangential component of **E** along the conductor; so V_{ab} (V at a due to I_b) is zero. By reciprocity, V_{ba} (V at b due to I_a) is zero. The terminals b are arbitrary; so the current element along the conductor (at a) produces no V between any two points in space; hence it produces no **E**. We can think of I_a as inducing currents on the conductor such that these currents produce a free-space field equal and opposite to the free-space field of I_a.

3-9. Green's Functions. Our reciprocity relationships are formulas symmetrical in two field-source pairs. Mathematical statements of reciprocity (symmetrical in two functions) are called *Green's theorems.* The difference between a Green's theorem and a reciprocity theorem is that no physical interpretation is given to the functions in the former.

The scalar Green's theorem is based on the identity

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla\psi \cdot \nabla\phi$$

When this is integrated throughout a region and the divergence theorem applied to the left-hand term, we obtain *Green's first identity*

$$\oint\!\!\!\!\oint \psi \frac{\partial \phi}{\partial n} ds = \iiint (\psi \nabla^2 \phi + \nabla\psi \cdot \nabla\phi)\, d\tau \tag{3-43}$$

Interchanging ψ and ϕ in this identity and subtracting the interchanged equation from the original equation, we obtain *Green's second identity* or *Green's theorem*

$$\oint\!\!\!\!\oint \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) ds = \iiint (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, d\tau \tag{3-44}$$

This is a statement of reciprocity for scalar fields ψ and ϕ.

The vector analogue to Green's theorem is based on the identity

$$\nabla \cdot (\mathbf{A} \times \nabla \times \mathbf{B}) = \nabla \times \mathbf{A} \cdot \nabla \times \mathbf{B} - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B}$$

An integration of this throughout a region and an application of the divergence theorem yields the vector analogue to Green's first identity

$$\oint\!\!\!\!\oint (\mathbf{A} \times \nabla \times \mathbf{B}) \cdot d\mathbf{s} = \iiint (\nabla \times \mathbf{A} \cdot \nabla \times \mathbf{B} - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B})\, d\tau \tag{3-45}$$

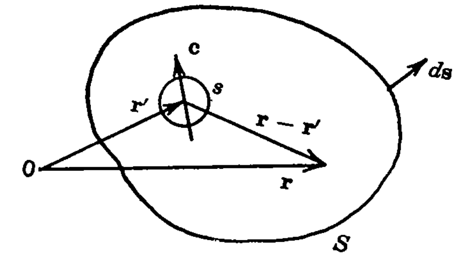

FIG. 3-20. Region to which Green's theorem is applied.

We can interchange **A** and **B** and subtract the resulting equation from the original equation. This gives the vector analogue to Green's second identity, or the vector Green's theorem,

$$\oint\!\!\!\!\oint (\mathbf{A} \times \nabla \times \mathbf{B} - \mathbf{B} \times \nabla \times \mathbf{A}) \cdot d\mathbf{s} = \iiint (\mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B})\, d\tau \tag{3-46}$$

Our reciprocity theorem [Eq. (3-35)], for a *homogeneous* medium, is essentially Eq. (3-46) with $\mathbf{A} = \mathbf{E}^a$ and $\mathbf{B} = \mathbf{E}^b$. For an inhomogeneous medium, still another vector Green's theorem corresponds to our reciprocity theorem (see Prob. 3-28).

Green's theorems have been used extensively in the literature as follows. Suppose we desire the field **E** at a point $\mathbf{r}'$ in a region. Instead of solving this problem directly, a point source is placed at $\mathbf{r}'$, and its field is called a *Green's function* **G**. We then substitute $\mathbf{E} = \mathbf{A}$ and $\mathbf{G} = \mathbf{B}$ in Eq. (3-46). This gives a formula for **E** at $\mathbf{r}'$, as we shall discuss below. What we have done is solve the reciprocal problem (source at the field point of the original problem) and then apply reciprocity. The equivalence principle gives the solution more directly.

Let us summarize the various Green's functions used in the literature. Stratton chooses[1]

$$\mathbf{G}_1 = \mathbf{c}\phi \tag{3-47}$$

where

$$\phi = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{3-48}$$

and **c** is a constant vector. A comparison of Eq. (3-47) with Eq. (2-117) shows that $\mathbf{G}_1$ is the vector potential of a current element $Il = 4\pi\mathbf{c}$. Hence, $\mathbf{G}_1$ is a solution to Eq. (2-108), or

$$\nabla \times \nabla \times \mathbf{G}_1 - k^2\mathbf{G}_1 = \nabla(\nabla \cdot \mathbf{G}_1) \qquad \mathbf{r} \neq \mathbf{r}' \tag{3-49}$$

Now suppose we wish to find **E** at $\mathbf{r}'$ in a source-free region enclosed by S. The source of $\mathbf{G}_1$ is placed at $\mathbf{r}'$ and surrounded by an infinitesimal sphere s, as shown in Fig. 3-20. Equation (3-46) with $\mathbf{A} = \mathbf{E}$ and $\mathbf{B} = \mathbf{G}_1$ is now

[1] J. A. Stratton, "Electromagnetic Theory," p. 464, McGraw-Hill Book Company, Inc., New York, 1941.

applied to the region enclosed by S and s. The result is

$$-4\pi\mathbf{c}\cdot\mathbf{E} = \oiint_S (\mathbf{E}\times\nabla\times\mathbf{G}_1 - \mathbf{G}_1\times\nabla\times\mathbf{E} + \mathbf{E}\,\nabla\cdot\mathbf{G}_1)\cdot d\mathbf{s} \qquad (3\text{-}50)$$

which is a formula for calculating $\mathbf{E}$ at $\mathbf{r}'$ in terms of $\mathbf{n}\times\mathbf{E}$, $\mathbf{n}\times\nabla\times\mathbf{E}$, and $\mathbf{n}\cdot\mathbf{E}$ on S. Furthermore, it is required that $\mathbf{E}$ be continuous and have continuous first derivatives on S. This is a severe restriction on the usefulness of Eq. (3-50), although it can be amended to admit singular $\mathbf{E}$'s on S.

A choice of Green's function which overcomes some of the disadvantages of Eq. (3-50) is[1]

$$\mathbf{G}_2 = \nabla\times\mathbf{c}\phi \qquad (3\text{-}51)$$

where ϕ is given by Eq. (3-48). This is evidently the magnetic field of a current element $Il = 4\pi\mathbf{c}$. Hence, $\mathbf{G}_2$ is a solution to

$$\nabla\times\nabla\times\mathbf{G}_2 - k^2\mathbf{G}_2 = 0 \qquad \mathbf{r}\neq\mathbf{r}' \qquad (3\text{-}52)$$

We now apply Eq. (3-46) with $\mathbf{A} = \mathbf{E}$ and $\mathbf{B} = \mathbf{G}_2$ to the region enclosed by S and s in Fig. 3-20. The result is[2]

$$4\pi\mathbf{c}\cdot\nabla'\times\mathbf{E} = \oiint_S (\mathbf{G}_2\times\nabla\times\mathbf{E} - \mathbf{E}\times\nabla\times\mathbf{G}_2)\cdot d\mathbf{s} \qquad (3\text{-}53)$$

This is a formula for $\nabla'\times\mathbf{E}$ (hence for $\mathbf{H}$) at $\mathbf{r}'$ in terms of $\mathbf{n}\times\mathbf{E}$ and $\mathbf{n}\times\nabla\times\mathbf{E}$ on S. Equation (3-53) does not require $\mathbf{E}$ to be continuous on S, nor do we need to know $\mathbf{n}\cdot\mathbf{E}$ on S. Thus, Eq. (3-53) is a substantial improvement over Eq. (3-50). In fact, Eq. (3-53) can be shown to be identical to the formula obtained from the equivalence principle of Fig. 3-9, applied to a homogeneous medium.

Another useful Green's function is

$$\mathbf{G}_3 = \nabla\times\nabla\times\mathbf{c}\phi \qquad (3\text{-}54)$$

where ϕ is given by Eq. (3-48). This is proportional to the electric field of an electric current element; so $\mathbf{G}_3$ also satisfies Eq. (3-52). An application of Eq. (3-46) would yield a formula for $\mathbf{E}$ at $\mathbf{r}'$, similar in form to Eq. (3-53).

All of the $\mathbf{G}$'s considered so far are "free-space" Green's functions, that is, they are fields of sources radiating into unbounded space. We can choose other $\mathbf{G}$'s such that they satisfy boundary conditions on S.

[1] J. R. Mentzer, "Scattering and Diffraction of Radio Waves," p. 14, Pergamon Press, New York, 1955.

[2] The left-hand side of this equation is a function only of the primed coordinates. Hence, a prime is placed on ∇' to indicate operation on $\mathbf{r}'$ instead of $\mathbf{r}$.

For example, let

$$\mathbf{G}_4 = \mathbf{G}_2 + \mathbf{G}_4^s \qquad (3\text{-}55)$$

such that $\mathbf{G}_4$ satisfies Eq. (3-52) and

$$\mathbf{n}\times\nabla\times\mathbf{G}_4 = 0 \qquad \text{on } S \qquad (3\text{-}56)$$

The physical interpretation of $\mathbf{G}_4$ is that it is the magnetic field of a current element $Il = 4\pi\mathbf{c}$ radiating in the presence of a perfect electric conductor over S. The $\mathbf{G}_2$ is the incident field, and the $\mathbf{G}_4^s$ is the scattered field. Application of Eq. (3-46) with $\mathbf{A} = \mathbf{E}$ and $\mathbf{B} = \mathbf{G}_4$ results in Eq. (3-53) with the last term zero, because of Eq. (3-56). Thus,

$$4\pi\mathbf{c}\cdot\nabla'\times\mathbf{E} = \oiint_S (\mathbf{G}_4\times\nabla\times\mathbf{E})\cdot d\mathbf{s} \qquad (3\text{-}57)$$

which is a formula for $\nabla'\times\mathbf{E}$ in terms of only $\mathbf{n}\times\nabla\times\mathbf{E}$ over S. The same formula can be obtained from the equivalence principle of Fig. 3-11, as it applies to a homogeneous region.

Similarly, defining a $\mathbf{G}_5$ such that

$$\mathbf{n}\times\mathbf{G}_5 = 0 \qquad \text{on } S \qquad (3\text{-}58)$$

we can obtain a formula

$$4\pi\mathbf{c}\cdot\nabla'\times\mathbf{E} = -\oiint_S (\mathbf{E}\times\nabla\times\mathbf{G}_5)\cdot d\mathbf{s} \qquad (3\text{-}59)$$

and so on. All these various formulas, and many more, can be directly obtained from the equivalence principle. We have discussed the Green's function approach merely because it has been used extensively in the literature.

3-10. Tensor Green's Functions. We shall henceforth use the term "Green's function" to mean "field of a point source." Suppose we have a current element Il at $\mathbf{r}'$ and we wish to evaluate the field $\mathbf{E}$ at $\mathbf{r}$. The most general linear relationship between two vector quantities can be represented by a tensor. Hence, the field $\mathbf{E}$ is related to the source Il by

$$\mathbf{E} = [\Gamma]Il \qquad (3\text{-}60)$$

where $[\Gamma]$ is called a *tensor Green's function.* In rectangular components and matrix notation, Eq. (3-60) becomes

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \Gamma_{xx} & \Gamma_{xy} & \Gamma_{xz} \\ \Gamma_{yx} & \Gamma_{yy} & \Gamma_{yz} \\ \Gamma_{zx} & \Gamma_{zy} & \Gamma_{zz} \end{bmatrix} \begin{bmatrix} Il_x \\ Il_y \\ Il_z \end{bmatrix} \qquad (3\text{-}61)$$

Thus, Γ_{ij} is the ith component of $\mathbf{E}$ due to a unit j-directed electric current element. The $\mathbf{E}$ might be the free-space field of Il, in which case

$[\Gamma]$ would be the "free-space Green's function." Alternatively, **E** might be the field of Il radiating in the presence of some matter, and $[\Gamma]$ would then be called the "Green's function subject to boundary conditions." Still other Green's functions are those relating **H** to Il, those relating **E** to Kl, and so on.

Our principal use of tensor Green's functions will be for concise mathematical expression. For example, the equation

$$\mathbf{E} = \iiint [\Gamma]\mathbf{J}\, d\tau' \tag{3-62}$$

where $[\Gamma]$ is the free-space Green's function defined by Eq. (3-60), represents the solution of Eq. (2-111), which is

$$\begin{aligned} \mathbf{E} &= -j\omega\mu\mathbf{A} + \frac{1}{j\omega\epsilon}\nabla(\nabla\cdot\mathbf{A}) \\ \mathbf{A} &= \iiint \frac{\mathbf{J}e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d\tau' \end{aligned} \tag{3-63}$$

Equation (3-62) also represents the field of currents in the vicinity of a material body if $[\Gamma]$ represents the appropriate Green's function, and so on. In other words, Eq. (3-62) is symbolic of the solution, regardless of whether or not we can find $[\Gamma]$.

Even though we shall not use tensor Green's functions to find explicit solutions, it should prove instructive to find an explicit $[\Gamma]$. Let us take $[\Gamma]$ to be the free-space Green's function defined by Eq. (3-60). If Il is x-directed,

$$A_x = \frac{Il e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

and

$$\begin{aligned} E_x &= -j\omega\mu A_x + \frac{1}{j\omega\epsilon}\frac{\partial^2 A_x}{\partial x^2} \\ E_y &= \frac{1}{j\omega\epsilon}\frac{\partial^2 A_x}{\partial y\,\partial x} \\ E_z &= \frac{1}{j\omega\epsilon}\frac{\partial^2 A_x}{\partial z\,\partial x} \end{aligned}$$

Comparing this with Eq. (3-61) for $Il_y = Il_z = 0$, we see that

$$\begin{aligned} \Gamma_{xx} &= \left(-j\omega\mu + \frac{1}{j\omega\epsilon}\frac{\partial^2}{\partial x^2}\right)\psi \\ \Gamma_{yx} &= \frac{1}{j\omega\epsilon}\frac{\partial^2\psi}{\partial y\,\partial x} \\ \Gamma_{zx} &= \frac{1}{j\omega\epsilon}\frac{\partial^2\psi}{\partial z\,\partial x} \end{aligned}$$

where

$$\psi = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{3-64}$$

The other elements of $[\Gamma]$ are found by taking Il to be y-directed and then z-directed. From symmetry considerations, the other Γ_{ij}'s will differ only by a cyclic interchange of (x,y,z). The result is therefore

$$\begin{aligned} \Gamma_{ii} &= \left(-j\omega\mu + \frac{1}{j\omega\epsilon}\frac{\partial^2}{\partial i^2}\right)\psi \\ \Gamma_{ij} &= \frac{1}{j\omega\epsilon}\frac{\partial^2\psi}{\partial i\,\partial j} \qquad i \neq j \end{aligned} \tag{3-65}$$

with ψ given by Eq. (3-64). The reciprocity theorem is reflected in the symmetry

$$\Gamma_{ij}(\mathbf{r},\mathbf{r}') = \Gamma_{ji}(\mathbf{r}',\mathbf{r}) \tag{3-66}$$

which can be proved for Γ's subject to boundary conditions as well.

3-11. Integral Equations. An integral equation is one for which the unknown quantity appears in an integrand. We already have the concepts needed to construct integral equations. For example, the potential integral of Eq. (2-118) is essentially an integral equation when **J** is unknown. Most problems can be formulated either in terms of integral equations or in terms of differential equations. When *exact* solutions are desired, the differential equation approach is usually the simpler one. An important use of integral equations is to obtain *approximate* solutions. There is good reason for this. Integration is a summation process, and it is not necessary that each element of the summation be correct. Errors in some elements of the summation may be compensated for by errors in other elements. Also, all elements do not contribute equally to a summation. It is much more important that the elements contributing most to the summation be correct than that the elements of minor contribution be correct. This is why we were able to obtain useful results by assuming the current on the linear antenna of Fig. 2-23, by assuming the field of each element of magnetic current in Fig. 3-17*b*, and so on.

To illustrate the formulation of an integral equation, consider the induction theorem of Fig. 3-16. Let $[\Gamma(\mathbf{r},\mathbf{r}')]$ be the tensor relating the **E** field at **r** due to an element of **M** at **r**′ radiating in the presence of the conductor over S. In equation form, this is

$$d\mathbf{E}(\mathbf{r}) = [\Gamma(\mathbf{r},\mathbf{r}')]\, d\mathbf{M}(\mathbf{r}')$$

The total scattered field for the problem is then the summation

$$\mathbf{E}^s(\mathbf{r}) = \oint\!\!\oint_S [\Gamma(\mathbf{r},\mathbf{r}')]\mathbf{M}_s(\mathbf{r}')\, ds'$$

where $\mathbf{M}_s$ is given by Eq. (3-27). When **r** is on S, Eq. (3-26) must also be true; hence

$$\mathbf{n} \times \mathbf{E}^i(\mathbf{r}) = \mathbf{n} \times \oiint_S [\Gamma(\mathbf{r},\mathbf{r}')]\mathbf{E}^i(\mathbf{r}') \times d\mathbf{s}' \qquad \mathbf{r} \text{ on } S \qquad (3\text{-}67)$$

The incident field $\mathbf{E}^i$ is assumed to be known; so Eq. (3-67) is an integral equation for determining $[\Gamma]$. As we mentioned earlier, an exact solution to Eq. (3-67) would be difficult even for the simplest specialization.

Problems involving a region homogeneous except for small "islands" of matter are commonly encountered. Examples of such problems are the linear antenna of Fig. 2-23 and the obstacle of Fig. 3-15*a*. To illustrate the general concepts involved, suppose we have an inhomogeneous region, possibly containing sources $\mathbf{J}^i$ and $\mathbf{M}^i$. Within this region, the field satisfies

$$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H} + \mathbf{M}^i \qquad \nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J}^i$$

where $\hat{z}$ and $\hat{y}$ are functions of position. We can define *normal* values of impedivity and admittivity, $\hat{z}_1$ and $\hat{y}_1$, which may be any convenient constants (usually the most common $\hat{z}$ and $\hat{y}$ in the region). We can now rewrite the field equations as

$$-\nabla \times \mathbf{E} = \hat{z}_1\mathbf{H} + \mathbf{M} \qquad \nabla \times \mathbf{H} = \hat{y}_1\mathbf{E} + \mathbf{J}$$

where the *effective* currents are

$$\begin{aligned} \mathbf{M} &= (\hat{z} - \hat{z}_1)\mathbf{H} + \mathbf{M}^i \\ \mathbf{J} &= (\hat{y} - \hat{y}_1)\mathbf{E} + \mathbf{J}^i \end{aligned} \qquad (3\text{-}68)$$

These effective currents can then be treated as source currents in a homogeneous region. Since $\mathbf{J}$ and $\mathbf{M}$ are functions of $\mathbf{E}$ and $\mathbf{H}$, a solution in terms of them will lead to an integral equation. However, if $\hat{z} = \hat{z}_1$ and $\hat{y} = \hat{y}_1$ except in small subregions, we can assume $\mathbf{J}$ and $\mathbf{M}$ in the subregions and obtain approximate expressions for $\mathbf{E}$ and $\mathbf{H}$ elsewhere. (Recall the linear antenna problem, where we assumed I on the antenna wire.) Note that, when the normal $\hat{z}$ and $\hat{y}$ are taken as the free-space parameters, Eqs. (3-68) reduce to

$$\begin{aligned} \mathbf{M} &= j\omega(\hat{\mu} - \mu_0)\mathbf{H} + \mathbf{M}^i \\ \mathbf{J} &= j\omega(\hat{\epsilon} - \epsilon_0)\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}^i \end{aligned} \qquad (3\text{-}69)$$

The effective currents in excess of the true sources ($\mathbf{M}^i$ and $\mathbf{J}^i$) are now just those due to the motion of atomic particles in vacuum.

Let us reconsider the problem of scattering by an obstacle in the light of the above concepts. Given the problem of Fig. 3-15*a*, we can consider the total field to be the potential integral solution of Eqs. (3-4) and (3-5), with $\mathbf{J}$ and $\mathbf{M}$ given by Eqs. (3-69). The incident field is that produced

by $\mathbf{J}^i$ and $\mathbf{M}^i$ outside of the obstacle, and the scattered field is that produced by

$$\begin{aligned} \mathbf{M} &= j\omega(\hat{\mu} - \mu_0)\mathbf{H} \\ \mathbf{J} &= j\omega(\hat{\epsilon} - \epsilon_0)\mathbf{E} + \sigma\mathbf{E} \end{aligned} \qquad (3\text{-}70)$$

throughout the obstacle. To be explicit, outside of the obstacle

$$\mathbf{E}^s = -\nabla \times \mathbf{F} + \frac{1}{j\omega\epsilon_0}\nabla \times \nabla \times \mathbf{A} \qquad (3\text{-}71)$$

where
$$\begin{aligned} \mathbf{A} &= \frac{1}{4\pi}\iiint_{\text{obstacle}} \frac{\mathbf{J}e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \\ \mathbf{F} &= \frac{1}{4\pi}\iiint_{\text{obstacle}} \frac{\mathbf{M}e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \end{aligned} \qquad (3\text{-}72)$$

with $\mathbf{J}$ and $\mathbf{M}$ given by Eq. (3-70). If we can guess $\mathbf{J}$ and $\mathbf{M}$ with reasonable accuracy, then Eqs. (3-71) and (3-72) will give us an approximate solution. For a nonmagnetic obstacle, $\mathbf{M}$, and consequently $\mathbf{F}$, will be zero. For a good conductor, $\mathbf{J}$ reduces to $\sigma\mathbf{E}$, and this current resides primarily on the surface of the obstacle. If we assume the obstacle perfectly conducting, then $\mathbf{J}$ becomes a true surface current. The solution in this case reduces to

$$\mathbf{E}^s = \frac{1}{4\pi j\omega\epsilon_0}\nabla \times \nabla \times \oiint_S \frac{\mathbf{J}_s e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,ds' \qquad (3\text{-}73)$$

If we specialize this equation to S, then Eq. (3-26) must be met, and we have an integral equation for determining $\mathbf{J}_s$.

An approximation to $\mathbf{J}_s$, known as the *physical optics approximation*, is as follows. Let Fig. 3-21*a* represent a perfectly conducting obstacle illuminated by some source. In terms of the total field, the surface current on the conductor is given by

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}$$

When the obstacle is large, we assume that the total field is negligible in

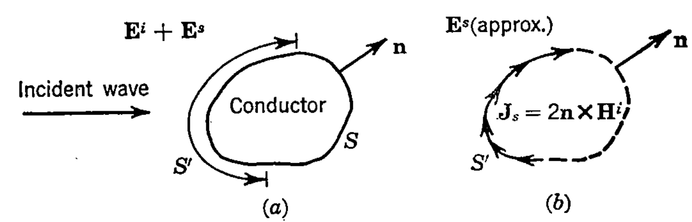

FIG. 3-21. The physical optics approximation. (*a*) Original problem; (*b*) the approximation.

the "shadow" region. Furthermore, if the obstacle is smooth and gently curved, each element of surface behaves similarly to an element of a ground plane. According to image theory, the tangential components of **H** at a ground plane are just twice those from the same source in unbounded space. We therefore approximate the current on the obstacle by

$$\mathbf{J}_s \approx 2\mathbf{n} \times \mathbf{H}^i \qquad \text{over } S' \tag{3-74}$$

where S' is the illuminated portion of S. The physical optics approximation to the scattered field is therefore

$$\mathbf{E}^s \approx \frac{1}{2\pi j\omega\epsilon_0} \nabla \times \nabla \times \iint_{S'} \frac{(\mathbf{n} \times \mathbf{H}^i) e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} ds' \tag{3-75}$$

This approximation is illustrated by Fig. 3-21*b*.

As an explicit application of the physical optics approximation, again consider the large conducting plate of Fig. 3-17*a*. The incident **E** is given by Eq. (3-28); hence

$$H_y{}^i = \frac{E_0}{\eta} e^{-jkz}$$

The physical optics approximation to the obstacle current [Eq. (3-74)] is therefore

$$J_x = \frac{2E_0}{\eta}$$

Each element of this radiates as a current element in free space, as analyzed in Sec. 2-9. The contribution to the radiation field in the back-scatter direction from each $J_x\, ds$ is

$$dE_x{}^s = \frac{-jkE_0\, ds}{2\pi r} e^{-jkr}$$

The total distant back-scattered field is therefore

$$E_x{}^s = \iint_{\text{plate}} dE_x{}^s = -\frac{jkE_0 A}{2\pi r} e^{-jkr} \tag{3-76}$$

which is identical to Eq. (3-29), the approximation obtained from the induction theorem. The physical optics approximation to the echo area of the plate is therefore that of Eq. (3-31). This equality of the two approximations to back scattering [Eqs. (3-29) and (3-76)] is no coincidence. It can be shown that the two approaches *always* give the same back scattering but *do not* give the same scattering in other directions.[1]

[1] R. F. Harrington, On Scattering by Large Conducting Bodies, *IRE Trans.*, vol. AP-7, no. 2, pp. 150–153, April, 1959.

3-12. Construction of Solutions. So far, we have explicitly considered only two types of solutions to the field equations, namely, uniform plane waves and the potential integrals. In the next three chapters, we shall learn how to construct many other solutions. A general method of obtaining these solutions is considered here.

In a homogeneous source-free region, the field satisfies

$$\begin{aligned} -\nabla \times \mathbf{E} &= \hat{z}\mathbf{H} \qquad & \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{H} &= \hat{y}\mathbf{E} \qquad & \nabla \cdot \mathbf{E} &= 0 \end{aligned} \tag{3-77}$$

In view of the divergenceless character of **E** and **H**, we can express the field in terms of a magnetic vector potential **A** or in terms of an electric vector potential **F**. More important, we can employ superposition and express part of the field in terms of **A** and part in terms of **F**. The **A** must be a solution to Eq. (2-108) with $\mathbf{J} = 0$, and the **F** a solution to the dual equation. The general equations for vector potentials are therefore

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} &= -\hat{y}\nabla\Phi^a \\ \nabla \times \nabla \times \mathbf{F} - k^2\mathbf{F} &= -\hat{z}\nabla\Phi^f \end{aligned} \tag{3-78}$$

where Φ^a and Φ^f are arbitrary scalars. The electromagnetic field in terms of **A** and **F** is given by Eqs. (3-4) with $\mathbf{J} = \mathbf{M} = 0$, or

$$\begin{aligned} \mathbf{E} &= -\nabla \times \mathbf{F} + \frac{1}{\hat{y}} \nabla \times \nabla \times \mathbf{A} \\ \mathbf{H} &= \nabla \times \mathbf{A} + \frac{1}{\hat{z}} \nabla \times \nabla \times \mathbf{F} \end{aligned} \tag{3-79}$$

Equations (3-78) and (3-79) are the general form for fields and potentials in homogeneous source-free regions.

There is a great deal of arbitrariness in the choice of vector potentials. For instance, we can choose the arbitrary Φ's according to

$$\nabla \cdot \mathbf{A} = -\hat{y}\Phi^a \qquad \nabla \cdot \mathbf{F} = -\hat{z}\Phi^f \tag{3-80}$$

This reduces Eqs. (3-78) to

$$\begin{aligned} \nabla^2\mathbf{A} + k^2\mathbf{A} &= 0 \\ \nabla^2\mathbf{F} + k^2\mathbf{F} &= 0 \end{aligned} \tag{3-81}$$

Solutions to these equations are called *wave potentials*. Note that the rectangular components of the wave potentials satisfy the scalar wave equation, or Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0 \tag{3-82}$$

Also, when Eqs. (3-80) are satisfied, we can alternatively write Eqs.

(3-79) as

$$\mathbf{E} = -\nabla \times \mathbf{F} - \hat{z}\mathbf{A} + \frac{1}{\hat{y}}\nabla(\nabla \cdot \mathbf{A})$$
$$\mathbf{H} = \nabla \times \mathbf{A} - \hat{y}\mathbf{F} + \frac{1}{\hat{z}}\nabla(\nabla \cdot \mathbf{F}) \tag{3-83}$$

We have yet to decide how to divide the field between **A** and **F**. As a word of caution, *do not* make the mistake of thinking of **A** as due to **J** and **F** as due to **M**. This happened to be our choice for the potential integral solution, where we considered the sources everywhere. We are now concerned with regions of finite extent, and we can represent a field in terms of **A** or **F** or both, regardless of its actual source.

Let us now consider some particular choices of potentials. If we take $\mathbf{F} = 0$ and

$$\mathbf{A} = \mathbf{u}_z\psi \tag{3-84}$$

then

$$\mathbf{E} = -\hat{z}\mathbf{A} + \frac{1}{\hat{y}}\nabla(\nabla \cdot \mathbf{A}) \qquad \mathbf{H} = \nabla \times \mathbf{A} \tag{3-85}$$

This can be expanded in rectangular coordinates as

$$E_x = \frac{1}{\hat{y}}\frac{\partial^2\psi}{\partial x\,\partial z} \qquad H_x = \frac{\partial\psi}{\partial y}$$
$$E_y = \frac{1}{\hat{y}}\frac{\partial^2\psi}{\partial y\,\partial z} \qquad H_y = -\frac{\partial\psi}{\partial x} \tag{3-86}$$
$$E_z = \frac{1}{\hat{y}}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi \qquad H_z = 0$$

A field with no H_z is called *transverse magnetic to z* (TM). We shall find it possible to choose ψ sufficiently general to express an arbitrary TM field in a homogeneous source-free region according to the above formulas.

In the dual sense, if we choose $\mathbf{A} = 0$ and

$$\mathbf{F} = \mathbf{u}_z\psi \tag{3-87}$$

then

$$\mathbf{E} = -\nabla \times \mathbf{F} \qquad \mathbf{H} = -\hat{y}\mathbf{F} + \frac{1}{\hat{z}}\nabla(\nabla \cdot \mathbf{F}) \tag{3-88}$$

Expanded in rectangular coordinates, this is

$$E_x = -\frac{\partial\psi}{\partial y} \qquad H_x = \frac{1}{\hat{z}}\frac{\partial^2\psi}{\partial x\,\partial z}$$
$$E_y = \frac{\partial\psi}{\partial x} \qquad H_y = \frac{1}{\hat{z}}\frac{\partial^2\psi}{\partial y\,\partial z} \tag{3-89}$$
$$E_z = 0 \qquad H_z = \frac{1}{\hat{z}}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

A field with no E_z is called *transverse electric to z* (TE). We shall find it possible to choose ψ sufficiently general to express any TE field in a homogeneous source-free region according to the above formulas.

Now suppose we have a field neither TE nor TM. We can determine a ψ according to

$$\frac{\partial^2\psi^a}{\partial z^2} + k^2\psi^a = \hat{y}E_z$$

which will generate a field TM to z according to Eqs. (3-86). This TM field will have the same E_z as does the original field; so the difference between the two will be a TE field. We can therefore determine this difference field according to Eqs. (3-89), where the ψ is found from

$$\frac{\partial^2\psi^f}{\partial z^2} + k^2\psi^f = \hat{z}H_z$$

Thus, *an arbitrary field in a homogeneous source-free region can be expressed as the sum of a* TM *field and a* TE *field.* Explicit expressions for the field would be superposition of Eqs. (3-86) and (3-89), with superscripts a and f added to the ψ's to distinguish between them. Since the z direction is arbitrary, we can express this independent of the coordinate system by defining

$$\mathbf{A} = \mathbf{c}\psi^a \qquad \mathbf{F} = \mathbf{c}\psi^f \tag{3-90}$$

where **c** is a *constant* vector. The field is then given by Eqs. (3-79), which become

$$\mathbf{E} = -\nabla \times (\mathbf{c}\psi^f) + \frac{1}{\hat{y}}\nabla \times \nabla \times (\mathbf{c}\psi^a)$$
$$\mathbf{H} = \nabla \times (\mathbf{c}\psi^a) + \frac{1}{\hat{z}}\nabla \times \nabla \times (\mathbf{c}\psi^f) \tag{3-91}$$

where the ψ's are solutions to Eq. (3-82). We must therefore study solutions to the scalar Helmholtz equation to learn how to pick the ψ's.

If the region is not source-free but is still homogeneous, our starting equations are

$$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H} + \mathbf{M}$$
$$\nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J} \tag{3-92}$$

instead of Eqs. (3-77). General solutions to Eqs. (3-92) can be constructed as the sum of any possible solution, called a *particular solution*, plus a solution to the source-free equations, called a *complementary solution*. We already have a particular solution, namely, the potential integral solution of Sec. 3-2. Therefore, solutions in a homogeneous region containing sources are given by

$$\mathbf{E} = \mathbf{E}_{ps} + \mathbf{E}_{cs} \qquad \mathbf{H} = \mathbf{H}_{ps} + \mathbf{H}_{cs} \tag{3-93}$$

where the particular solution (ps) is formed according to Eqs. (3-4) and (3-5), and the complementary solution (cs) is constructed according to Eqs. (3-91). We can think of the particular solution as the field due to sources inside the region and the complementary solution as the field due to sources outside the region.

3-13. The Radiation Field. It is easier to evaluate the radiation (distant) field from sources of finite extent than to evaluate the near field. (See, for example, Secs. 2-9 and 2-10.) In this section, we shall formalize the procedure for specializing solutions to the radiation zone.

Consider a distribution of currents in the vicinity of the coordinate origin, immersed in a homogeneous region of infinite extent. The complete solution to the problem is represented by Eqs. (3-4) and (3-5). If we specialize to the radiation zone ($r \gg r'_{\max}$), as suggested by Fig. 3-22, we have

$$|\mathbf{r} - \mathbf{r}'| \to r - r' \cos \xi \qquad (3\text{-}94)$$

where ξ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Furthermore, the second term of Eq. (3-94) can be neglected in the "magnitude factors," $|\mathbf{r} - \mathbf{r}'|^{-1}$, of Eqs. (3-5). It cannot, however, be neglected in the "phase factors," $\exp(-jk|\mathbf{r} - \mathbf{r}'|)$, unless $r'_{\max} \ll \lambda$. Thus, Eqs. (3-5) reduce to

$$\begin{aligned} \mathbf{A} &= \frac{e^{-jkr}}{4\pi r} \iiint \mathbf{J}(\mathbf{r}') e^{jkr' \cos \xi}\, d\tau' \\ \mathbf{F} &= \frac{e^{-jkr}}{4\pi r} \iiint \mathbf{M}(\mathbf{r}') e^{jkr' \cos \xi}\, d\tau' \end{aligned} \qquad (3\text{-}95)$$

in the radiation zone. Note that we now have the r dependence shown explicitly. Many of the operations of Eqs. (3-4) can therefore be performed.

Rather than blindly expanding Eqs. (3-4), let us draw upon some previous conclusions. In Sec. 2-9 it was shown that the distant field of an electric current element was essentially outward-traveling plane waves. The same is true of a magnetic current element, by duality. Hence, the

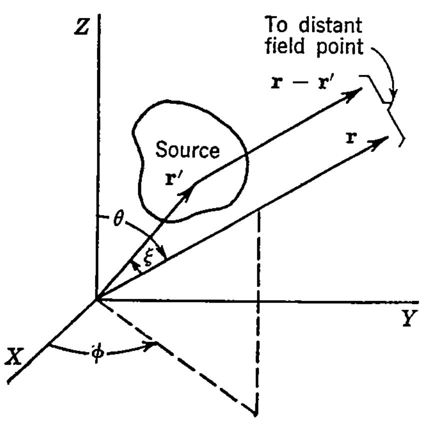

FIG. 3-22. Geometry for evaluating the radiation field.

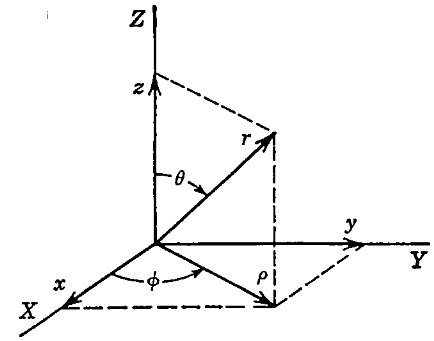

FIG. 3-23. Conventional coordinate orientation.

radiation zone must be characterized by

$$E_\theta = \eta H_\phi \qquad E_\phi = -\eta H_\theta \qquad (3\text{-}96)$$

since it is a superposition of the fields from many current elements. We can evaluate the partial $\mathbf{H}$ field due to $\mathbf{J}$ according to $\mathbf{H}' = \nabla \times \mathbf{A}$ (see Sec. 3-2). Retaining only the dominant terms (r^{-1} variation), we have

$$\begin{aligned} H'_\theta &= (\nabla \times \mathbf{A})_\theta = jkA_\phi \\ H'_\phi &= (\nabla \times \mathbf{A})_\phi = -jkA_\theta \end{aligned}$$

with $\mathbf{E}'$ given by Eqs. (3-96). Similarly, for the partial $\mathbf{E}$ field due to $\mathbf{M}$ we have, in the radiation zone,

$$\begin{aligned} E''_\theta &= -(\nabla \times \mathbf{F})_\theta = -jkF_\phi \\ E''_\phi &= -(\nabla \times \mathbf{F})_\phi = jkF_\theta \end{aligned}$$

with $\mathbf{H}''$ given by Eqs. (3-96). The total field is the sum of these partial fields, or

$$\begin{aligned} E_\theta &= -j\omega\mu A_\theta - jkF_\phi \\ E_\phi &= -j\omega\mu A_\phi + jkF_\theta \end{aligned} \qquad (3\text{-}97)$$

in the radiation zone, with $\mathbf{H}$ given by Eqs. (3-96). Thus, no differentiation of the vector potentials is necessary to obtain the radiation field.

Also, for future reference, let us determine $r' \cos \xi$ as a function of the source coordinates. The three coordinate systems of primary interest are the rectangular, cylindrical, and spherical, as illustrated by Fig. 3-23. For the conventional orientation shown, we have the transformations

$$\begin{aligned} x &= r \sin\theta \cos\phi & x &= \rho \cos\phi \\ y &= r \sin\theta \sin\phi & y &= \rho \sin\phi \\ z &= r \cos\theta & z &= z \end{aligned} \qquad (3\text{-}98)$$

To obtain $r' \cos \xi$, we form

$$rr' \cos \xi = \mathbf{r} \cdot \mathbf{r}' = xx' + yy' + zz' \qquad (3\text{-}99)$$

Substituting for x, y, z from the first set of Eqs. (3-98), we obtain

$$r' \cos \xi = (x' \cos \phi + y' \sin \phi) \sin \theta + z' \cos \theta \qquad (3\text{-}100)$$

which is the desired form when rectangular coordinates are chosen for the source. Substituting into Eq. (3-100) for x', y', z' from the second set of Eqs. (3-98), we obtain

$$r' \cos \xi = \rho' \sin \theta \cos (\phi - \phi') + z' \cos \theta \qquad (3\text{-}101)$$

which is the desired form when cylindrical coordinates are chosen for the source. Finally, substituting into Eq. (3-100) for x', y', z' from the first set of Eqs. (3-98), we have

$$r' \cos \xi = r'[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')] \qquad (3\text{-}102)$$

which is the desired form when spherical coordinates are chosen for the source.

PROBLEMS

3-1. Show that a current sheet

$$\mathbf{J} = \mathbf{u}_x J_0$$

over the $z = 0$ plane produces the outward-traveling plane waves

$$E_x = \begin{cases} -\dfrac{\eta J_0}{2} e^{-jkz} & z > 0 \\ -\dfrac{\eta J_0}{2} e^{jkz} & z < 0 \end{cases}$$

in an infinite homogeneous medium.

3-2. Instead of the electric current sheet, suppose that the magnetic current sheet

$$\mathbf{M}_s = \mathbf{u}_y M_0 \sin \frac{\pi y}{b}$$

exists over the cross section $z = 0$ in the waveguide of Fig. 3-2. Show that this magnetic current produces a field

$$E_x = \begin{cases} -\dfrac{M_0}{2} \sin \dfrac{\pi y}{b} e^{-j\beta z} & z > 0 \\ \dfrac{M_0}{2} \sin \dfrac{\pi y}{b} e^{j\beta z} & z < 0 \end{cases}$$

3-3. Suppose now that the two current sheets

$$\mathbf{J}_s = \mathbf{u}_x \frac{A}{Z_0} \sin \frac{\pi y}{b}$$
$$\mathbf{M}_s = \mathbf{u}_y A \sin \frac{\pi y}{b}$$

exist simultaneously over the cross section $z = 0$ of Fig. 3-2. Show that these produce a field

$$E_x = \begin{cases} -A \sin \dfrac{\pi y}{b} e^{-j\beta z} & z > 0 \\ 0 & z < 0 \end{cases}$$

This source is a "directional coupler."

3-4. In Fig. 3-2, suppose that a "shorting plate" (conductor) is placed over the cross section $z = -d$. Show that the current sheet of Eq. (3-2) now produces a field

$$E_x = \begin{cases} -\dfrac{J_0 Z_0}{2} (1 - e^{-j2\beta d}) \sin \dfrac{\pi y}{b} e^{-j\beta z} & z > 0 \\ -jJ_0 Z_0 e^{-j\beta d} \sin \dfrac{\pi y}{b} \sin [\beta(d + z)] & -d < z < 0 \end{cases}$$

Note that when d is an odd number of guide quarter-wavelengths, E_x for $z > 0$ is twice that for the current sheet alone [see Eq. (3-3)], but when d is an integral number of guide half-wavelengths, no E_x exists for $z > 0$.

3-5. The TE and TM modes of a parallel-plate waveguide (Prob. 2-28) are almost dual to each other. Show that the field dual to the TE_n mode of Prob. 2-28 is the TM_n mode for the parallel-plate guide having conductors over the planes $y = b/2$ and $y = -b/2$. Show that the field dual to the TM_n mode of Prob. 2-28 is the TE_n mode of this new waveguide.

3-6. Obtain the field of an infinitesimal loop of magnetic current having z-directed moment KS. Show that this produces the same field as the electric current element of Fig. 2-21 if

$$Il = -j\omega\epsilon KS$$

3-7. Figure 3-24a shows the cross section of a "twin-slot" transmission line. Show that the field distribution is dual to that of the collinear plate line of Fig. 3-24b. By integrating along the contours shown in Fig. 3-24c, determine the line voltages and

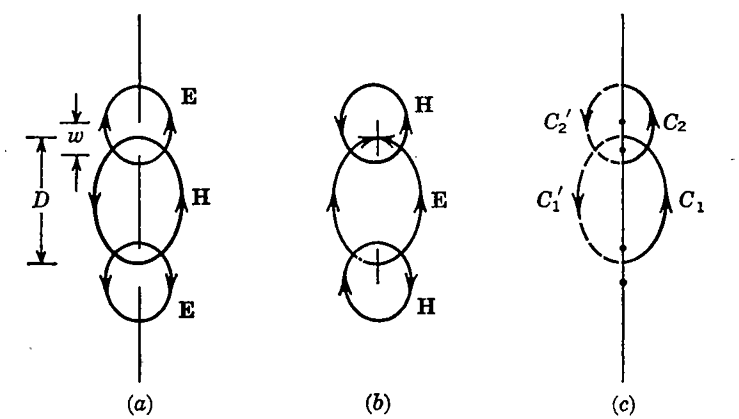

FIG. 3-24. Figures for Prob. 3-7. (a) Twin-slot line; (b) collinear plate line; (c) integration contours.

currents of both the slot line and the plate line. Show that

$$(Z_0)_{\text{slot line}} = \frac{\eta^2}{4(Z_0)_{\text{plate line}}}$$

From Table 2-3, it follows that

$$(Z_0)_{\text{slot line}} \approx \frac{\eta\pi}{4 \log (4D/w)} \qquad D \gg w$$

The two transmission lines are said to be complementary structures (see Babinet's principle, Sec. 7-12).

3-8. Show that the field

$$E_x = \begin{cases} \dfrac{J_0 Z_0}{2} \sin \dfrac{\pi y}{b}\, e^{j\beta z} & z > 0 \\ \dfrac{J_0 Z_0}{2} \sin \dfrac{\pi y}{b}\, e^{-j\beta z} & z < 0 \end{cases}$$

is also a mathematical solution to the problem of Fig. 3-2 with $\mathbf{J}_s$ given by Eq. (3-2). What do our uniqueness theorems say about this second solution? What can we say about it on physical grounds? Give a couple of other possible solutions to the problem, and interpret them physically.

3-9. Show that the current sheets

$$\mathbf{J}_s = -\mathbf{u}_\theta \frac{Il}{4\pi} e^{-jka} \left(\frac{jk}{a} + \frac{1}{a^2} \right) \sin\theta$$

$$\mathbf{M}_s = -\mathbf{u}_\phi \frac{Il}{4\pi} e^{-jka} \left(\frac{j\omega\mu}{a} + \frac{\eta}{a^2} + \frac{1}{j\omega\epsilon a^3} \right) \sin\theta$$

over the sphere $r = a$ produce the field of Eqs. (2-113) $r > a$ and zero field $r < a$.

3-10. If $\mathbf{E}$ is well-behaved in a homogeneous region bounded by S, and if $\hat{z}\mathbf{H} = -\nabla \times \mathbf{E}$, show that the currents

$$\mathbf{J} = -\hat{y}\mathbf{E} - \frac{1}{\hat{z}} \nabla \times \nabla \times \mathbf{E}$$

will support this and only this field among a class $\mathbf{E}$, $\mathbf{H}$ having identical tangential components of $\mathbf{E}$ on S. Show that the same $\mathbf{E}$, but different $\mathbf{H}$, can be obtained within this class if magnetic sources $\mathbf{K}$ are allowed in addition to $\mathbf{J}$.

3-11. Suppose there exists within the rectangular cavity of Fig. 2-19 a field

$$E_x = E_0 \sin \frac{\pi y}{b} \sinh \gamma z$$

where $\gamma = \sqrt{(\pi/b)^2 - k^2}$ and k is complex (lossy dielectric). Show that this field can be supported by the source

$$\mathbf{M}_s = -\mathbf{u}_y E_0 \sin \frac{\pi y}{b} \sinh \gamma c$$

at the wall $z = c$. Show that for a low-loss dielectric, $\mathbf{M}_s$ almost vanishes at the resonant frequency [Eq. (2-95)], that is, a small $\mathbf{M}_s$ produces a large $\mathbf{E}$.

3-12. Consider a z-directed current element Il a distance d in front of a ground plane covering the $y = 0$ plane, as shown in Fig. 3-25. Show that the radiation field is given by

$$E_\theta = \frac{-\eta Il}{\lambda r} e^{-jkr} \sin\theta \sin (kd \sin\phi \sin\theta)$$

and $\eta H_\phi = E_\theta$. Find the power radiated and show that the radiation resistance referred to I is

$$R_r = \frac{\eta\pi l^2}{\lambda^2} \left[\frac{2}{3} - \frac{\sin 2kd}{2kd} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3} \right]$$

For $d \le \lambda/4$, the maximum radiation is in the y direction. Show that

$$R_r \xrightarrow[kd \to 0]{} \eta \frac{32\pi^3 l^2 d^2}{15\lambda^4}$$

and that the gain is 7.5 for d small, 4.15 for $d = \lambda/4$, and approximately 6 for d large.

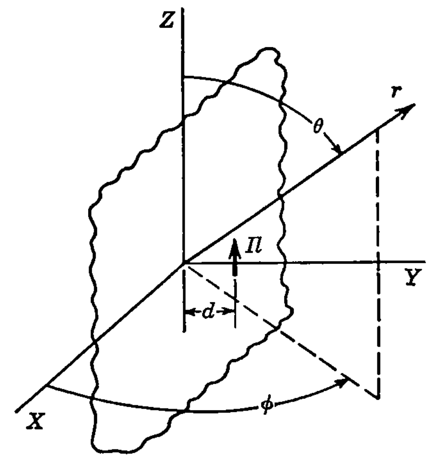

FIG. 3-25. Current element parallel to a ground plane.

3-13. In Fig. 3-6a, suppose we have a small loop of electric current with z-directed moment IS, instead of the current element. Show that the radiation field is given by

$$E_\phi = \frac{j\eta 2\pi IS}{\lambda^2 r} e^{-jkr} \sin (kd \cos\theta) \sin\theta$$

and $\eta H_\theta = -E_\phi$. Find the power radiated and show that the radiation resistance referred to I is

$$R_r = 2\pi\eta \left(\frac{kS}{\lambda} \right)^2 \left[\frac{1}{3} + \frac{\cos 2kd}{(2kd)^2} - \frac{\sin 2kd}{(2kd)^3} \right]$$

For small d,

$$E_\phi \xrightarrow[kd \to 0]{} \frac{j\eta\pi ISkd}{\lambda^2 r} e^{-jkr} \sin 2\theta$$

$$R_r \xrightarrow[kd \to 0]{} \frac{\pi\eta}{15} \left(\frac{kSkd}{\lambda} \right)^2$$

Thus, maximum radiation is at $\theta = 45°$ for small d. The gain at small d is 15. For large d, the maximum radiation lies close to the ground plane, and the gain is 6.

3-14. In Fig. 3-25, suppose we have a small loop of electric current with z-directed moment IS, instead of the current element. Show that the radiation field is given by

$$E_\phi = \frac{\eta k^2 IS}{2\pi r} e^{-jkr} \sin\theta \cos (kd \sin\phi \sin\theta)$$

and $\eta H_\theta = -E_\phi$. Show that the radiation resistance referred to I is

$$R_r = \pi\eta\left(\frac{kS}{\lambda}\right)^2\left[\frac{2}{3} + \frac{\sin 2kd}{2kd} + \frac{\cos 2kd}{(2kd)^2} - \frac{\sin 2kd}{(2kd)^3}\right]$$

The maximum radiation is along the ground plane, in the x direction. For small kd,

$$R_r \xrightarrow[kd\to 0]{} \frac{4\pi\eta}{3}\left(\frac{kS}{\lambda}\right)^2$$

which is twice that for the isolated loop. For $d = 0$, the gain is 3; for $d = \lambda/4$, it is 7.1; and for $d \to \infty$, it is 6.

3-15. The monopole antenna consists of a straight wire perpendicular to a ground plane, fed at the ground plane, as shown in Fig. 3-26. Show that the field is the same as that from the dipole antenna (Fig. 2-23), fed at the center. Show that the gain of the monopole is twice that of the corresponding dipole and that the radiation resistance is one-half. For example, the radiation resistance of the $\lambda/4$ monopole is 36.6 ohms.

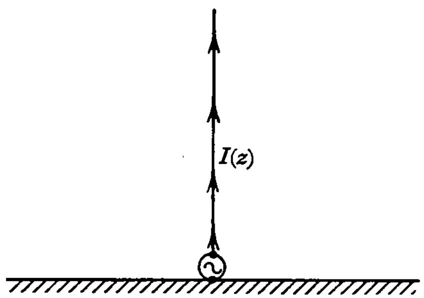

FIG. 3-26. The monopole antenna.

3-16. Consider an open-ended coaxial line (Fig. 3-14a without the ground plane) of small radii a and b. Treat the problem according to the equivalence principle as applied to a surface just enclosing the coax. Assume $\mathbf{n} \times \mathbf{H}$ is essentially zero over the entire surface and that tangential $\mathbf{E}$ is that of the transmission-line mode over the open end. Show that to this approximation the radiated field is one-half that of Eq. (3-20) and that the radiation conductance is one-half that of Eq. (3-23).

3-17. A slot antenna consists of a slot in a conducting ground plane, as shown in Fig. 3-27. It is called a dipole slot antenna when fed by a voltage impressed across the center of the slot. The slot and ground plane can be viewed as a transmission line, and the field in the slot will be essentially a harmonic function of kz. Assume

$$E_x = \frac{V_m}{w}\sin\left[k\left(\frac{L}{2} - |z|\right)\right]$$

in the slot, and obtain the magnetic current equivalent of the form of Fig. 3-13c. For w small, show that this equivalent representation is the dual problem to the dipole antenna of Sec. 2-10. Using duality, show that the radiation field is

$$\frac{jV_m e^{-jkr}}{\eta\pi r}\,\frac{\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)}{\sin\theta} = \begin{cases} H_\theta & y > 0 \\ -H_\theta & y < 0 \end{cases}$$

Define the radiation conductance of this antenna as $G_r = \mathcal{P}_f/|V_m|^2$, and show that

$$(G_r)_{\text{slot dipole}} = \frac{4(R_r)_{\text{wire dipole}}}{\eta^2}$$

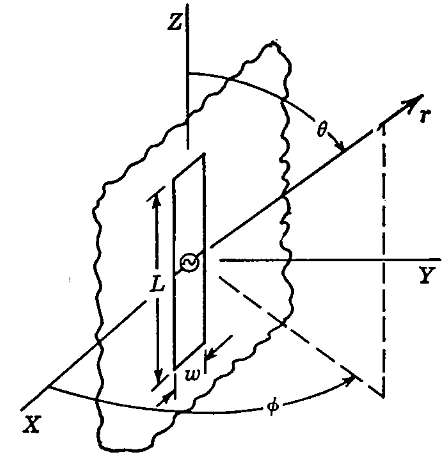

FIG. 3-27. A slot antenna.

where R_r is as plotted in Fig. 2-24. The input voltage V_i is related to V_m by $V_i = V_m \sin(kL/2)$; so the input conductance is given by

$$G_i = \frac{G_r}{\sin^2\left(k\frac{L}{2}\right)}$$

3-18. For the antenna of Fig. 3-27, assume E_x in the slot the same as in Prob. 3-17, and show that for arbitrary w

$$\frac{jV_m e^{-jkr}}{\eta\pi r} f(\theta,\phi) = \begin{cases} H_\theta & y > 0 \\ -H_\theta & y < 0 \end{cases}$$

where
$$f(\theta,\phi) = \frac{\sin\left(k\frac{w}{2}\cos\phi\sin\theta\right)}{k\frac{w}{2}\cos\phi\sin\theta}\left[\frac{\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)}{\sin\theta}\right]$$

3-19. Figure 3-28 shows an aperture antenna consisting of a rectangular waveguide opening onto a ground plane. Assume that E_x in the aperture is that of the TE_{01}

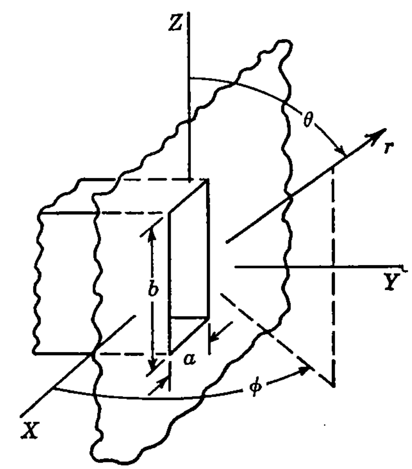

FIG. 3-28. A rectangular waveguide opening onto a ground plane.

waveguide mode, and show that the radiation field is

$$H_\theta = \frac{2jbE_0e^{-jkr}}{\eta r} \frac{\sin\left(k\frac{a}{2}\cos\phi\sin\theta\right)\cos\left(k\frac{b}{2}\cos\theta\right)}{\cos\phi\,[\pi^2 - (kb\cos\theta)^2]}$$

3-20. Figure 3-29 represents a rectangular conducting plate of width a in the y direction and b in the z direction. Let the incident plane wave be specified by

$$E_z^i = E_0e^{jk(x\cos\phi_0 + y\sin\phi_0)}$$

Use the induction theorem with the same approximation as was used in the problem

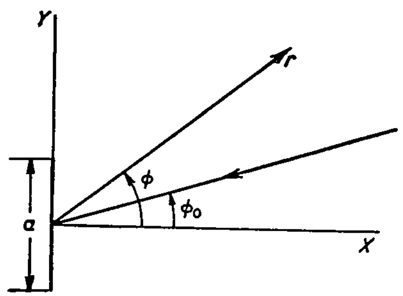

FIG. 3-29. Scattering by a rectangular plate.

of Fig. 3-17, and show that at large r the scattered field in the xy plane is

$$E_z^s \approx \frac{kE_0abe^{-jkr}}{j2\pi r}\frac{\sin\,[k(a/2)(\sin\phi + \sin\phi_0)]}{k(a/2)(\sin\phi + \sin\phi_0)}\cos\phi$$

Show that the echo area is

$$A_e \approx 4\pi\left[\frac{ab\cos\phi_0\sin\,(ka\sin\phi_0)}{\lambda ka\sin\phi_0}\right]^2$$

3-21. Repeat Prob. 3-20 for the orthogonal polarization, that is,

$$H_z^i = H_0e^{jk(x\cos\phi_0 + y\sin\phi_0)}$$

and show that at large r the scattered field in the xy plane is

$$H_z^s \approx \frac{jkH_0abe^{-jkr}}{2\pi r}\frac{\sin\,[k(a/2)(\sin\phi + \sin\phi_0)]}{k(a/2)(\sin\phi + \sin\phi_0)}\cos\phi_0$$

Show that the echo area is the same as obtained in Prob. 3-20.

3-22. Use reciprocity to evaluate the radiation field of the dipole antenna of Sec. 2-10. To do this, place a θ-directed current element at large r, and apply Eq. (3-36), obtaining Eq. (2-125).

3-23. By applying voltage sources to the network of Fig. 3-18, show that the admittance matrix $[y]$ defined by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

satisfies the reciprocity relationship $y_{12} = y_{21}$ when Eq. (3-38) is valid.

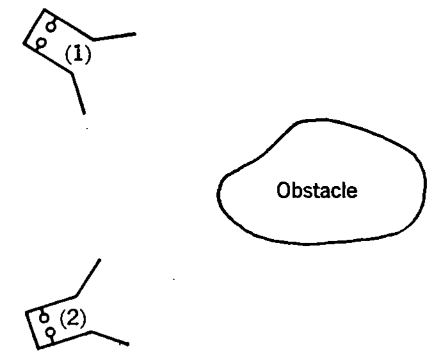

FIG. 3-30. Differential scattering.

3-24. Let Fig. 3-30 represent two antennas in the presence of an obstacle. Let V_1 be the voltage received at antenna 1 when a unit current source is applied at antenna 2 and V_2 be the voltage received at antenna 2 when a unit current source is applied at antenna 1. Let V_1^i and V_2^i be the corresponding voltages when the obstacle is absent. Define the scattered voltages as

$$V_1^s = V_1 - V_1^i \qquad V_2^s = V_2 - V_2^i$$

and show that $V_1^s = V_2^s$.

3-25. For the problem of Fig. 3-2, define the input impedance of the sheet of current as

$$Z = -\frac{\langle a,a\rangle}{I^2}$$

where $\langle a,a\rangle$ is the self-reaction of the currents and I is the total current of the sheet. Evaluate Z when the field is given by Eqs. (3-3).

3-26. Repeat Prob. 3-25 for the current sheet and field of Prob. 3-4.

3-27. In the vector Green's theorem [Eq. (3-46)], let $\mathbf{A} = \mathbf{E}^a$ and $\mathbf{B} = \mathbf{E}^b$ in a homogeneous isotropic region, and show that it reduces to Eq. (3-35).

3-28. Use the vector identity

$$\nabla\cdot(\mathbf{A}\times\phi\nabla\times\mathbf{B}) = \phi\nabla\times\mathbf{A}\cdot\nabla\times\mathbf{B} - \mathbf{A}\cdot\nabla\times\phi\nabla\times\mathbf{B}$$

and derive the modified vector Green's theorem

$$\oint\!\!\oint \phi(\mathbf{A}\times\nabla\times\mathbf{B} - \mathbf{B}\times\nabla\times\mathbf{A})\cdot d\mathbf{s} = \iiint (\mathbf{B}\cdot\nabla\times\phi\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\phi\nabla\times\mathbf{B})\,d\tau$$

Let $\mathbf{A} = \mathbf{E}^a$, $\mathbf{B} = \mathbf{E}^b$, $\phi = \hat{z}^{-1}$ in an inhomogeneous region, and show that the above theorem reduces to Eq. (3-35).

3-29. Derive the left-hand term of Eq. (3-50), that is, show

$$\oint\!\!\oint_s (\mathbf{E}\times\nabla\times\mathbf{G}_1 - \mathbf{G}_1\times\nabla\times\mathbf{E} + \mathbf{E}\,\nabla\cdot\mathbf{G}_1)\cdot d\mathbf{s} \xrightarrow[|\mathbf{r}-\mathbf{r}'|\to 0]{} 4\pi\mathbf{c}\cdot\mathbf{E}$$

3-30. Let $\mathbf{G}_4$ be the magnetic field of a z-directed current element situated $y > 0$ and radiating in the presence of a perfect electric conductor covering the $y = 0$ plane. In other words, let $\mathbf{c} = \mathbf{u}_z$ and S be the $y = 0$ plane. Show that

$$\mathbf{G}_4 = \nabla \times \mathbf{u}_z \left(\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2} \right)$$

where

$$r_1 = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$
$$r_2 = \sqrt{(x - x')^2 + (y + y')^2 + (z - z')^2}$$

3-31. Specialize the $\mathbf{G}_4$ of Prob. 3-30 to $r_1 \to \infty$, and apply Eq. (3-57) to the problem of Fig. 3-28. Show that this gives the same answer as obtained in Prob. 3-19.

3-32. Apply duality to Eqs. (3-65), and evaluate the magnetic tensor Green's function $[\Gamma]$ defined by

$$\mathbf{H} = [\Gamma]Kl$$

in free space.

3-33. Evaluate the Γ_{ij} for the free-space tensor Green's function defined by

$$\mathbf{H} = [\Gamma]Il$$

3-34. Repeat Prob. 3-20 using the physical optics approximation, and show that the answer for E_z^s differs from that of Prob. 3-20 by an interchange of ϕ and ϕ_0. Show that the echo area is identical to that of Prob. 3-20.

3-35. Repeat Prob. 3-21 using the physical optics approximation, and show that the answer for H_z^s differs from that of Prob. 3-21 by an interchange of ϕ and ϕ_0. Show that the echo area is identical to that of Prob. 3-21.

3-36. Let $\psi = e^{-jky}$ in Eqs. (3-86), and evaluate the electromagnetic field. Classify this field in as many ways as you can (wave-type, polarization, etc.).

3-37. Let $\psi = e^{-jkx}$ in Eqs. (3-89), and evaluate the electromagnetic field. Classify this field in as many ways as you can.

3-38. Let $\mathbf{c} = \mathbf{u}_x$, $\psi^a = e^{-jkz}$, $\psi^f = je^{-jkz}$, and evaluate Eqs. (3-91). Classify this field in as many ways as you can.

3-39. Derive Eqs. (3-97) by expanding Eqs. (3-4) with **A** and **F** as given by Eqs. (3-95).

CHAPTER 4

PLANE WAVE FUNCTIONS

4-1. The Wave Functions. The problems that we have considered so far are of two types: (1) those reducible to sources in an unbounded homogeneous region, and (2) those solvable by using one or more uniform plane waves. Equations (3-91) show us how to construct general solutions to the field equations in homogeneous regions once we have general solutions to the scalar Helmholtz equation. By a method called *separation of variables*, general solutions to the Helmholtz equation can be constructed in certain coordinate systems.[1] In this section, we use the method of separation of variables to obtain solutions for the rectangular coordinate system.

The Helmholtz equation in rectangular coordinates is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \tag{4-1}$$

The method of separation of variables seeks to find solutions of the form

$$\psi = X(x)Y(y)Z(z) \tag{4-2}$$

that is, solutions which are the product of three functions of one coordinate each. Substitution of Eq. (4-2) into Eq. (4-1), and division by ψ, yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0 \tag{4-3}$$

Each term can depend, at most, on only one coordinate. Since each coordinate can be varied independently, Eq. (4-3) can sum to zero for all coordinate values only if each term is independent of x, y, and z. Thus, let

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2$$

where k_x, k_y, and k_z are constants, that is, are independent of x, y, and z. (The choice of minus a constant squared is taken for later convenience.)

[1] It has been shown by Eisenhart (*Ann. Math.*, vol. 35, p. 284, 1934) that the Helmholtz equation is separable in 11 three-dimensional orthogonal coordinate systems.